Evolution of Dishonest Behavior in Public Procurement. The Role of Updating Control

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Abstract

The audit level plays a crucial role in the prevalence of illegality in public procurement, specifically focusing on fraud. The aim of this paper is to investigate whether a forward-looking mechanism for updating the monitoring level by the State may influence the dishonest behavior in the economy and in which measure it depends on the spread of society's inherent honesty. With this aim in mind, we describe a model in which the monitoring level put in place by the State to fight non-compliant behavior depends on both the variation of the spread of dishonesty in the economy and on the previous auditing level, while considering economies at different levels of *honesty propensities*, i.e. greater inherent honesty stems from the stronger social disapproval of dishonesty. By combining analytical tools and numerical experiments, our model describes how the evolutionary adaptation process determines whether compliant or non-compliant behavior prevails in society. The main findings consist of: (1) a slight change in the monitoring function can influence significantly the asymptotic dynamics of the final map; (2) the effectiveness of public policies to combat illegality in public procurement depends on the spread of society's inherent honesty; (3) complex qualitative dynamics associated, in particular, with border collision bifurcations may emerge.

Keywords: Complex dynamics, Coexisting attractors, Public procurement, Dishonest behavior, Border collision bifurcations, Updating monitoring level

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1 Introduction

The public procurement serves as a mechanism through which a government acquires goods, services and work necessary for fulfilling their functions and delivering public services to citizens. Public procurement encompasses various activities, including tendering, contracting, purchasing and supplier management, all aimed at ensuring transparency, fairness and efficiency in the expenditure of public funds. At its core, public procurement plays a crucial role in promoting economic development and fostering competition within markets. By providing opportunities for businesses, including small and medium-sized enterprises (SMEs) to participate in government contracts, it stimulates innovation, drives economic growth, and enhances productivity. Additionally, public procurement serves as a mechanism for implementing government policies and priorities. Whether in infrastructure development, healthcare provision, education services, or defense procurement, it enables governments to translate their strategic objectives into tangible outcomes by selecting suppliers that align with their goals and objectives. Furthermore, public procurement contributes to the efficient allocation of resources by ensuring that goods and services are acquired at competitive prices and in accordance with quality standards, thus maximizing value for taxpayers' money.

Transparent and accountable procurement processes help prevent corruption, fraud, and favoritism, thereby safeguarding public trust and confidence in government institutions. In fact, "*Every year, over 250 000 public authorities in the EU spend around 14% of GDP (around €2 trillion per year) on the purchase of services, works and supplies. In many sectors such as energy, transport, waste management, social protection and the provision of health or education services, public authorities are the principal buyers. The public sector can use procurement to boost jobs, growth and investment, and to create an economy that is more innovative, resource and energy efficient, and socially-inclusive. High quality public services depend on modern, well-managed and efficient procurement*".[1]

[1] https://single-market-economy.ec.europa.eu/single-market/public-procurement_en

It is then critically important to enhance our comprehension of the illegality risks within public procurement. With public procurement reaching such high thresholds and expected to rise significantly in the years ahead (European Commission, 2020), literature suggests a noteworthy correlation between heightened government expenditure and corruption [1]. Given the current era of increased government spending and outsourcing to rejuvenate the post-Covid economy, the risk of illegality may escalate considerably. In fact, the complexity, scale, and interface between the public and private sectors make public procurement highly vulnerable to illegal practices. The significant contract values make procurement opportunities lucrative for bidders, fostering a willingness to engage in corrupt activities to secure contracts. Simultaneously, public officials may anticipate substantial gains from accepting bribes or kickbacks. Additionally, the intricate procedural stages of procurement offer multiple paths for corruption and enable the concealment of illicit activities. The close collaboration between public officials and private entities further exacerbates the potential for illegality, facilitating the exploitation of public power for private gain.

With high incentives for illegality and low risks of detection, public procurement remains exceedingly susceptible to illegal practices. Therefore, transparent procedures and efficient organizational frameworks are indispensable for fostering integrity within procurement systems. Thus, it is imperative to determine the effectiveness of measures aimed at combating corruption and fraud in public procurement. Efforts to combat fraud and corruption in public procurement typically involve implementing robust regulatory frameworks, enhancing transparency and accountability measures, strengthening oversight and enforcement mechanisms, promoting competition, and fostering a culture of integrity and ethical behavior within public institutions. This may include measures such as conducting due diligence on bidders, implementing anti-corruption compliance programs, ensuring competitive bidding processes, and establishing mechanisms for reporting and investigating allegations of misconduct.

Analyzing an endogenous monitoring technology tied to the spread of illegal behavior, we examine how the internal audit level impacts equilibrium within the public procurement sector when considering fraud. In our dynamic model, we delve into how the government conducts auctions for the supply of goods. Based on studies by [2], as well as [3], challenges arise due to discrepancies in the quality of goods, particularly when companies falsify original quality claims. Given that product quality is confidential, only public controllers can verify authenticity, thereby enabling the State to mitigate or eliminate illegal practices. In our model we extend the analysis done in [4] on the mechanism of the illegality control process, considering that the level of monitoring put in place by the State depends on both the variation spread of illegality in the economy and the previous level of monitoring, according to a forward-looking mechanism. This consideration stems from the realization that there is an "inertia" in the expenditure items of the public budget, since some of them refer to multi-year expenditure commitments that cannot be completely changed from one year to the next. In fact, regarding the level of control put in place in a period, this can be varied (decreased or increased) starting from the existing level of audit. In addition, it is relevant in deciding the future level of monitoring to consider the rate of change of illegality in the economy so as to put in place an effective strategy in combating illegality. Our model scrutinizes how the evolutionary adaptation process determines whether compliant or non-compliant behavior prevails in society. Players exhibit either compliant or non-compliant behavior depending on the type of firm encountered and the associated rewards. Firms exchange information through a word-of-mouth process, due to which interacting firms of different types may alter their behavior if the gains from the other approach outweigh those from their current choice.

However, we introduce the concept of *honesty propensity*: greater inherent honesty stems from the stronger social disapproval of dishonesty, as highlighted in the study by [5], emphasizing the significance of a country's "culture of legality". This inherent honesty is a factor that a government can influence, but only over the medium to long term. By examining and contrasting countries with varying attitudes towards honesty, we can assess how much the effectiveness of economic policies in combating dishonesty relies on initial conditions and the prevailing attitude of the country in question. From this assumption derives that wherein all dishonest firms interacting with honest ones choose to change behavior only if the expected utility from honesty exceeds that of dishonesty. Conversely, not all honest firms interacting with dishonest ones opt to change behavior, even if higher expected utility is attainable. Moreover, even in the case where being dishonest is more profitable, some corrupted firms may change their mind if interacting with honest firms. Notably, the evolutionary mechanism introduced here is asymmetric[2].

The economic framework is formalized through a discrete-time two-dimensional (2D) nonsmooth dynamical system (a map) delineating the evolution of both the fraction of dishonest firms and the monitoring level by the State over time. On the one hand, asymptotic dynamics of nonsmooth maps is known to be richer than dynamics of smooth ones. The presence of switching manifolds, which separate the state space into the regions where the map is defined differently, implies a new kind of bifurcation, called a *border collision bifurcation* (BCB). Collision of an invariant set with a switching manifold may cause an abrupt change in the phase portrait. A great number of works are devoted to studying BCBs and the induced phenomena ([7–10] to cite a few); in particular, to describing the related bifurcation structures observed in the parameter space of a map (see [11] and references therein). On the other hand, the investigated map also demonstrates other complexities in asymptotic behavior, including multistability (coexistence of several *nontrivial* attractors, often with a tangled basin structure) and an uncountable number of fixed points. The latter occurrence is related to the word-of-mouth mechanism that drives the evolution of the fraction of dishonest firms.

In what follows, two versions of the map are considered, which differ from the forward-looking mechanism for updating the State's monitoring level. In the first version, this mechanism is based on the *absolute* growth of the dishonest firms fraction. In this scenario qualitative dynamics are simple: any orbit asymptotically approaches one of the fixed points, which are infinite in number. Here the inner honesty plays a crucial role in the capability of the State to reduce non-compliant behavior, since in the case of an irresponsible society, only a sufficiently large level of monitoring can essentially reduce non-compliant behavior. However, strengthening the punishment (the fine) for dishonest behavior can increase the probability to concur frauds as well.

In the second version of the map, the monitoring level update mechanism is based on the *relative* variation of the dishonest firms fraction. In this case, high inner honesty does not help, in general, to

[2] A country's level of honesty develops over time when individual decision-making is subject to the influence of the group or others in society. For an application to tax compliance, using an Ising Model, see e.g. [6]

eliminate non-compliant behavior. Surprisingly though, in a society with a moderate level of endogenous dishonesty, provided that the initial fraction of dishonest firms is not too large, a sufficiently intensive monitoring can drive the system to a solution with all agents being honest. In this case, one can consider a restriction of the 2D map to the 1D manifold associated with the overall honesty. Furthermore, the final form of this 1D restriction has significant influence on the shape of 2D attractors that are different from fixed points. Nontrivial dynamics can emerge due to both, smooth bifurcations (such as fold, flip, etc.) and BCBs, causing complex bifurcation structures to occur in the parameter space of the 2D map, including regions of coexistence of several nontrivial attractors. Such an occurrence implies essential uncertainty, especially if the basins of some attractors are riddled, so that making prevision about the final outcome of the economy is almost impossible. Hence, policy suggestions are difficult to be given and setting the fine level to reduce dishonest behavior becomes a challenging choice.

The paper is organized as follows. In Section 2 we describe the model setup. In Section 3 we introduce the forward-looking updating control mechanism. In Section 4 we consider an evolution adaptation mechanism based on absolute variation. In Section 5 we consider an updating forward-looking mechanism based on relative variation. Section 6 concludes the paper.

2 Ingredients

We consider an economy composed of three types of risk-neutral players: the State, bureaucrats and firms (the number is normalized to one) and we assume that the State procures a unit of public good from each private firm in order to provide it free. Since the public good can be produced at different quality levels (low and high), even the government requires a high-quality public good and a firm could lie to the authorities regarding the quality. In such a case dishonest behavior in public procurement emerges. Following previous works such as [4] and [12], we consider a discrete time setup, i.e., $t = 0, 1, 2, ...$, and define $x_t \in [0, 1]$ as the fraction of firms producing low-level public goods who lie about the quality (dishonest firms) at time t.

As in [13] we assume that, at any time t, the price of public goods is constant and given by $p > 0$ while the per-unit production cost is c^h or c^l depending on the public good's quality (high or low respectively). Finally, the production of the public good is assumed to be profitable, and hence,

$$p > c^h > c^l > 0. \tag{1}$$

The State monitors the non-compliant behavior in public procurement. Let $q_t \in [0, 1]$ be the probability, at any time t, of being monitored according to the control level fixed by the State and, then, of being reported. If a dishonest firm is monitored and, hence, detected, it is punished with a constant fine $f > 0$.

Taking into account the previous considerations, at any time t, the expected utility of an honest firm per unit of procured public goods is given by

$$E[U_{h,t}] = U_h = p - c^h, \tag{2}$$

whereas the utility at time t of a dishonest firm depends on the event of being discovered with dishonest behavior, i.e.,

$$U_{d,t} = \begin{cases} U_{d,NM,t} = p - c^l, & \text{if not monitored,} \\ U_{d,M,t} = p - c^l - f, & \text{if monitored.} \end{cases} \tag{3}$$

Since the monitoring level may change at any time t, the expected utility at time t for a dishonest firm is given by

$$E[U_{d,t}] = q_t U_{d,M,t} + (1 - q_t) U_{d,NM,t} = p - c^l - f q_t. \tag{4}$$

The difference in expected utilities between dishonest and honest firms is then given by

$$\delta(q_t) = E[U_{d,t}] - U_h = \Delta_c - f q_t, \tag{5}$$

where $\Delta_c = c^h - c^l > 0$. Notice that $\delta(q_t)$ is a linear strictly decreasing function of the monitoring level, i.e., the difference between expected utilities decreases as the monitoring level increases, while the fine level affects its strength.

We define

$$\bar{q} = \frac{\Delta_c}{f} > 0 \; : \quad \delta(\bar{q}) = 0, \tag{6}$$

then $\bar{q}$ represents the monitoring level such that the two behaviors (honest and dishonest) results to be indifferent as they produce the same expected payoffs. Two cases may occur as described below.

(LF) A *low fine* case: $\bar{q} \geq 1$, i.e. the difference between production costs is higher than the fine and $\delta(q_t) \geq 0 \; \forall q_t \in [0, 1]$.

(HF) A *high fine* case: $\bar{q} < 1$, i.e. the difference between production costs is less than the fine, so that $\delta(q_t) \geq 0 \; \forall q_t \in [0, \bar{q}]$ while $\delta(q_t) < 0 \; \forall q_t \in (\bar{q}, 1]$. In such a case the difference between expected payoffs may be both positive or negative depending on the monitoring level fixed by the State at any given time.

In order to describe how dishonest behavior evolves over time, we consider a word-of-mouth mechanism as firstly proposed by [14] and [15]. In this evolutionary process, agents have the opportunity to compare their expected payoffs with those of others in society. If a firm encounters another firm exhibiting the same behavior (honest or dishonest), it gains no new insights into potential payoffs and thus decides to maintain its current behavior. Conversely, a firm may opt to change its behavior (from dishonest to honest, or vice versa) upon encountering a firm of a different type. After comparing their expected utilities, if the firm finds that switching types could increase its own expected utility, it may choose to transition from one type to the other. Then we follow the formalization given by [16] assuming that an honest firm encountering a dishonest one may alter its behavior if the payoff resulting from dishonest conduct surpasses that derived from honest behavior and vice versa. Then the equation describing the evolution of the fraction of dishonest firms over time is given by

$$x_{t+1} = F(x_t, q_t) = x_t[1 + (1 - x_t)(2\phi(\delta(q_t)) - 1)], \tag{7}$$

where $\phi(\delta(q_t))$ represents the probability for a single firm to switch from honest to dishonest behavior and it is described by a non-decreasing function $\phi : \mathbb{R} \to [0, 1]$ depending on $\delta(q_t)$. Note that, the probability for making the opposite change (switching from dishonest to honest) is then $1 - \phi$.

In order to specify function ϕ, we consider the *honesty propensity assumption* as firstly proposed in [3]. Hence, ϕ is formalized by the following continuous, increasing and piecewise smooth function:

$$\phi(\delta(q_t)) = \begin{cases} \phi_1(\delta(q_t)) = 1 - \frac{1}{\alpha\delta(q_t)+1}, & \text{if } \delta(q_t) \geq 0, \\ \\ \phi_2(\delta(q_t)) = 0, & \text{if } \delta(q_t) < 0, \end{cases} \tag{8}$$

where the parameter $\alpha > 0$ measures the *propensity to become dishonest* characterizing the country. In fact, we consider that the process guiding the transition of firms from one category to another is asymmetrical. Specifically, when dishonest firms encounter honest ones, they will opt to become honest if the benefits of honest behavior are at least equal to those of dishonest conduct. However, when honest firms encounter dishonest ones, only a portion will switch categories, even if this change could potentially lead to greater benefits. Thus, our model reflects an inherent inclination towards honesty whose strength negatively correlated with α.

3 Updating control

In order to complete the model, the evolution of the monitoring level put in place by the State to fight non-compliant behavior must be described. Notice that in previous works such as [3] and [4] it has been assumed that the State fixes the monitoring level to be set at time $t + 1$ by observing the fraction of dishonest firms at time t. In more detail, the following function has been considered

$$q_{t+1} = \gamma x_t^{\beta}, \quad \gamma \in (0, 1], \quad \beta > 0, \tag{9}$$

meaning that the State increases the audit efforts as the dishonesty level increases. At the same time, this formulation considers the existence of a maximum control level that can be reached, related both to the effort level the State wants to put in fighting dishonest behavior and/or to the existence of some budget constraints, i.e., a bound in the maximum amount of resources the State can use to fight dishonest behavior.

We modify the formulation proposed in (9) by making two improvements that make the hypotheses more closely match what can be found in real cases. The first point which we modify is the assumption that the monitoring level decided by the State to be fixed at time $t+1$ only depends on x_t, i.e., on the fraction of dishonest firms at time t. Such an assumption does not fully take into account that the monitoring level at time t influences the monitoring level the State is able to set at time $t+1$. In fact monitoring activities require human and monetary resources that cannot be completely renewed from one period to another. This consideration stems from the realization that there is an "inertia" in the expenditure items of the public budget since some of them refer to multi-year expenditure commitments that cannot be completely changed from one year to the next. In fact, regarding the level of control put in place in a period, this can be varied (decreased or increased) starting from the existing level of audit anyway. In addition, it is relevant in deciding the future level of monitoring to consider the rate of change of illegality in the economy so as to put in place an effective strategy in combating illegality. As a consequence, being q_t the monitoring level at time t, the State can only revise it from time t to time $t+1$, this means that q_{t+1} will depend on both variables, i.e., the observed dishonesty level x_t and the previous control level q_t.

The second point we improve is that in the previous works it is assumed that the State, when fixing the monitoring level to be reached at time $t+1$, considers the value of x_t, i.e. the dishonest behavior observed in the last period. Thus the expected value about the dishonest behavior emerging in the system is given by $E(x_{t+1}) = x_t$, revealing some kind of myopic expectation of the State on non-compliant behavior. In a different scenario, one can assume rational expectations, that is, ex-post, dishonest behavior can be observable by the government and, in addition, the State knows the future fraction x_{t+1} of dishonest firms in the economy thus setting q_{t+1} taking this knowledge into account. In any case, what we still consider is that there exists a constant $\gamma \in (0, 1]$ related to the budget constraint or to the maximum amount of resources that can be devoted to fighting dishonest behavior.

To consider the previous arguments the updating control strategy under rational expectations is introduced in the present work. Hence, the updating forward-looking monitoring function can be defined as

$$q_{t+1} = G(X_t, q_t), \tag{10}$$

where X_t depends proportionally on the difference $x_{t+1} - x_t$, that is, $X_t > 0$ if from the time t to $t+1$ the fraction of the dishonest firms increases, $X_t = 0$ in case $x_{t+1} = x_t$, and $X_t < 0$ otherwise. Then the monitoring level fixed at time $t+1$ is revised from the one fixed at time t by taking into account X_t. Since there exists an upper bound on the maximum effort put in place by the State to fight dishonest behavior, given by $\gamma \in (0, 1]$, the function G should satisfy the following properties:

(i) if $X_t > 0$ (dishonesty spreads), then q_t is increased until it reaches the maximum monitoring level γ depending on the budget constraints;

(ii) if $X_t < 0$ (non-compliant behavior reduces), then the monitoring level q_t is decreased but it should always remain non-negative;

(iii) if $X_t = 0$ then $q_{t+1} = q_t$, i.e. if the dishonest level does not change from one period to another, then there is no reason to change the monitoring level.

We introduce additionally a parameter $d \in (0, 1)$ related to the intensity with which q_t can grow/decline from time t to time $t+1$ in response to higher/lower observed dishonesty levels. More precisely, provided that q_t is given, the higher d values are associated with higher growth in the monitoring level by the State in response to an increased level of non-compliant behavior. In other words, the parameter d measures the capacity of the State to increase the monitoring level when more firms behave in a dishonest way and it depends on the strength of the budget constraints, on the monitoring technology, as well as on the labor market conditions.

Going back to the definition of G, a very simple function which verifies the properties (i)-(iii) is the following:

$$q_{t+1} = G(X_t, q_t) = \min\left\{q_t\left(1 + dX_t\right), \gamma\right\}, \tag{11}$$

provided that $X_t \in [-1, 1]$.

In order to study in depth the dynamics of dishonest behavior under the forward-looking updating control level, we need to specify X_t. In the following, we will discuss two different mechanisms, i.e. updating based on absolute variation and updating based on relative variation.

4 Updating based on absolute variation

The simplest and the most intuitive way to define X_t, is to equate it to the *absolute* growth of the fraction of dishonest firms, namely

$$X_t = x_{t+1} - x_t. \tag{12}$$

Recalling that x_{t+1} is given in (7), we get

$$X_t = X_t(x_t, q_t) = x_{t+1} - x_t = x_t(1 - x_t)(2\phi(\delta(q_t)) - 1),$$

and consequently, the final law describing the evolution of the monitoring level over time can be written as

$$q_{t+1} = G(x_t, q_t) = \min \left\{ q_t \left(1 + d x_t (1 - x_t)(2\phi(\delta(q_t)) - 1)\right), \gamma \right\}. \tag{13}$$

4.1 The dynamical system

Taking into account equations (7), (8) and (13) the final dynamical system $(x_t, q_t) \to S(x_t, q_t)$ can be obtained. As mentioned in Section 2, we distinguish between two cases: the low fine (LF) case with $\bar{q} = \Delta_c/f \geq 1$ and the high fine (HF) case with $\bar{q} < 1$. In the LF case, the evolution of the fraction x_t of dishonest firms and the monitoring level q_t by the State is described by the 2D piecewise smooth map $S_{LF} : K \to K$, $K = [0, 1] \times [0, \gamma]$, such that

$$S_{LF}(x_t, q_t) := \begin{cases} F_1(x_t, q_t) = x_t \left[1 + (1 - x_t) \frac{\alpha(\Delta_c - f q_t) - 1}{\alpha(\Delta_c - f q_t) + 1}\right], \\ \\ G_1(x_t, q_t) = \min \left\{ q_t \left(1 + d x_t (1 - x_t) \frac{\alpha(\Delta_c - f q_t) - 1}{\alpha(\Delta c - f q_t) + 1}\right), \gamma \right\}. \end{cases} \tag{14}$$

In the HF case, the evolution of x_t and q_t is described by the map $S_{HF} : K \to K$, where

$$S_{HF}(x_t, q_t) = \begin{cases} S_{LF}(x_t, q_t), & \text{if } q_t \in [0, \bar{q}], \\ \\ \bar{S}(x_t, q_t) = \begin{cases} F_2(x_t, q_t) = x_t^2, \\ \\ G_2(x_t, q_t) = \min \left\{ q_t \left(1 + d(x_t^2 - x_t)\right), \gamma \right\}, \end{cases} & \text{if } q_t \in (\bar{q}, \gamma]. \end{cases} \tag{15}$$

Finally, we define the general setup as

$$S(x_t, q_t) = \begin{cases} S_{LF}(x_t, q_t), & \text{if } \bar{q} \geq 1, \\ S_{HF}(x_t, q_t), & \text{if } \bar{q} < 1. \end{cases} \tag{16}$$

Note that the map S_{LF} represents a branch of the map S_{HF} and the analysis of its dynamics is included in the analysis of S_{HF}. Hence, it is enough to consider only the case $\bar{q} < 1$. From now on we assume that $\Delta_c < f$.

4.2 Equilibria and stability

The major particularity of the map S, given in (16), consists in the existence of an uncountable number of fixed points. ~~Namely, we have the following conditions for the fixed points with $(x, q) \in K$ (we omit the lower index $_t$ for the sake of brevity).~~ In this respect, the main facts can be summarized in the following Proposition.

Proposition 4.1. *The map S can have fixed points of three types:*

1. *For any $q \in [0, \gamma]$ the point $E_0^q(0, q)$ is a fixed point.*
2. *For any $q \in [0, \gamma]$ the point $E_1^q(1, q)$ is a fixed point.*
3. *For any $x \in (0, 1)$ and*

$$q^* = \frac{\alpha\Delta_c - 1}{\alpha f}, \tag{17}$$

the point $E_x^{q^*}(x, q^*)$ is a fixed point. Moreover, the line $\{(x, q^*) : 0 \leq x \leq 1\}$ belongs to the domain of the map $S_{LF} = (F_1, G_1)$.

Proof. The proof of items 1. and 2. is straightforward. For proving 3. we equate

$$x = F_1(x, q) \quad \text{and} \quad q = G_1(x, q),$$

which implies (17). The point $E_x^{q^*}$ belongs to the domain of S_{LF} if

$$q^* < \bar{q} \quad \Leftrightarrow \quad \frac{\alpha\Delta_c - 1}{\alpha f} < \frac{\Delta_c}{f}.$$

The latter always holds, since all parameters are positive. □

Note that the points E_0^q constitute the left boundary of the definition square K. We refer to them as "good" equilibria, since the number of dishonest firms becomes zero. Similalrly, the points E_1^q constitute the right boundary of K and are referred to as "bad" equilibria, because every firm becomes dishonest. The points $E_x^{q^*}$ represent the internal equilibria.

The existence of infinitely many "good" and "bad" equilibria is related to the word-of-mouth mechanism that drives the evolution of the fraction of dishonest firms in our model. In fact, as long as the firms are all honest or all dishonest, there is no way to change type, since no different information can be shared with another firm.

The uncountable number of internal fixed points with $q = q^*$ is explained by the form of the function ϕ defining the probability for a firm to switch from honest to dishonest behavior, namely, $\phi(\delta(q^*)) = 1/2$. In such a way, the number of honest firms choosing fraudulence equals the number of cheaters getting on the right path, and hence, the overall fraction of dishonest firms remains unchanged.

Clearly, for every fixed point mentioned above, inside its arbitrarily small neighborhood there is an uncountable number of other fixed points. Hence, every such fixed point is always locally stable in one direction, vertical (for E_0^q/E_1^q) or horizontal (for $E_x^{q^*}$). In the other direction these points can be stable or unstable, depending on the parameters and the location of the point itself.

Points $E_0^q(0,q)$: "good" equilibria.

Let us consider points E_0^q composing the left boundary of the definition square K. The Jacobian matrix evaluated in such a fixed point is

$$J_1\left(E_0^q\right) = \begin{pmatrix} -\dfrac{2}{\alpha(\Delta_c - fq)+1} + 2 & 0 \\ d\dfrac{\alpha(\Delta_c - fq)-1}{\alpha(\Delta_c - fq)+1}q & 1 \end{pmatrix}, \quad \text{if} \quad q \leq \bar{q}, \tag{18}$$

or

$$J_2\left(E_0^q\right) = \begin{pmatrix} 0 & 0 \\ -qd & 1 \end{pmatrix}, \quad \text{if} \quad q > \bar{q}. \tag{19}$$

One of the eigenvalues, related to the eigenvector $v_2(0,1)$ (vertical direction), is always $\lambda_2 = 1$. The other eigenvalue, related to the eigenvector $v_1(1,0)$ (horizontal direction), is

$$\lambda_1 = \begin{cases} -\dfrac{2}{\alpha(\Delta_c - fq) + 1} + 2 & \text{if} \quad q \leq \bar{q}, \\ 0 & \text{if} \quad q > \bar{q}. \end{cases} \tag{20}$$

Hence, the point E_0^q with $q > \bar{q}$ is always stable along the horizontal eigenvector.

Let us consider $q \leq \bar{q}$. The nontrivial eigenvalue is $\lambda_1 > -1$ if

$$q < \breve{q} = \frac{3\alpha\Delta_c + 1}{3\alpha f}. \tag{21}$$

Since $\breve{q} > \bar{q}$, there is always $\lambda_1 > -1$ for $q \leq \bar{q}$. Further, $\lambda_1 > 1$ if $q < q^*$, $\lambda_1 < 1$ if $q > q^*$, and $\lambda_1 = 1$ if $q = q^*$. ~~Consequently,~~ Thus, we have proved

Proposition 4.2. *A point E_0^q is*

(A.i) unstable along the eigenvector $v_1(1,0)$ if $q \in [0, q^*)$;

(A.ii) stable along v_1 if $q \in (q^*, \gamma]$.

Points $E_1^q(1,q)$: "bad" equilibria.

Let us now consider points E_1^q composing the right boundary of the definition square K. The Jacobian matrix evaluated in such a point is

$$J_1\left(E_1^q\right) = \begin{pmatrix} \dfrac{2}{\alpha(\Delta_c - fq) + 1} & 0 \\ -d\dfrac{\alpha(\Delta_c - fq) - 1}{\alpha(\Delta_c - fq) + 1}q & 1 \end{pmatrix}, \quad \text{if} \quad q \leq \bar{q}, \tag{22}$$

or

$$J_2\left(E_0^q\right) = \begin{pmatrix} 2 & 0 \\ qd & 1 \end{pmatrix}, \quad \text{if} \quad q > \bar{q}. \tag{23}$$

Again, the eigenvalue related to the vertical eigenvector $v_2(0,1)$ is $\lambda_2 = 1$. The other eigenvalue, related to the horizontal eigenvector $v_1(1,0)$, is

$$\lambda_1 = \begin{cases} \dfrac{2}{\alpha(\Delta_c - fq) + 1} & \text{if} \quad q \leq \bar{q}, \\ 2 & \text{if} \quad q > \bar{q}. \end{cases} \tag{24}$$

Hence, the point E_1^q with $q > \bar{q}$ is always unstable along the horizontal eigenvector.

For $q \leq \bar{q}$, the nontrivial eigenvalue is $\lambda_1 > -1$ if

$$q < \hat{q} = \frac{\alpha\Delta_c + 3}{\alpha f}. \tag{25}$$

There is $\hat{q} > \bar{q}$ and therefore $\lambda_1 > -1$ for $q \leq \bar{q}$. For the other stability condition there holds $\lambda_1 < 1$ if $q < q^*$, $\lambda_1 > 1$ if $q > q^*$, and $\lambda_1 = 1$ if $q = q^*$. In such a way, there holds

Proposition 4.3. *A point E_1^q is*

(B.i) stable along the eigenvector $v_1(1,0)$ if $q \in [0, q^*)$;

(B.ii) unstable along v_1 if $q \in (q^*, \gamma]$.

Points $E_x^{q^*}(x, q^*)$: internal equilibria.

Finally we consider fixed points $E_x^{q^*}(x, q^*)$, which belong to the definition square K only if $\alpha\Delta_c > 1$. The respective Jacobian is

$$J^* = \begin{pmatrix} 1 & -\dfrac{\alpha f x(1-x)}{2} \\ 0 & 1 - \dfrac{(\alpha\Delta_c - 1)dx(1-x)}{2} \end{pmatrix} \tag{26}$$

and its eigenvalues are $\lambda_1 = 1$ related to the horizontal eigenvector $v_1(1,0)$, along which the point is locally stable, and

$$\lambda_2 = 1 - \frac{(\alpha\Delta_c - 1)dx(1-x)}{2} \tag{27}$$

related to the vertical eigenvector $v_2(0,1)$. Concerning the stability along v_2, we can state that $\lambda_2 < 1$ for any $x \in (0,1)$ (for $\alpha\Delta_c > 1$). Additionally, $\lambda_2 > -1$ if

$$x(1-x) < \frac{4}{(\alpha\Delta_c - 1)d}. \tag{28}$$

Since $\max x(1-x) = 1/4$, for $(\alpha\Delta_c - 1)d < 16$, there is $\lambda_2 > -1$ for any $x \in (0,1)$. If $(\alpha\Delta_c - 1)d \geq 16$, there exist

$$0 < \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{4}{(\alpha\Delta_c - 1)d}} = x_1 \leq x_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4}{(\alpha\Delta_c - 1)d}} < 1, \tag{29}$$

such that for $x \in (0, x_1)$ and $x \in (x_2, 1)$ there is $\lambda_2 > -1$, while it is $\lambda_2 < -1$ for $x \in (x_1, x_2)$. Finally, $\lambda_2 = -1$ for $x = x_1$ or $x = x_2$. In such a way we ~~get, that~~ can formulate the following

Proposition 4.4. *For the internal equilibria there hold:*

(C.i) If $\alpha\Delta_c \leq 1$, the fixed points $E_x^{q^*}$, $x \in (0,1)$, are located outside the definition square K;[3]

(C.ii) If $\alpha\Delta_c > 1$ and $(\alpha\Delta_c - 1)d < 16$, fixed points $E_x^{q^*}$, $x \in (0,1)$, are stable along the eigenvector $v_2(0,1)$;

(C.iii) If $(\alpha\Delta_c - 1)d \geq 16$, only fixed points $E_x^{q^*}$ for $x \in (0, x_1)$ and $x \in (x_2, 1)$, with x_1 and x_2 given by (29), are stable along v_2.

For all fixed points considered above, stability along the respective direction (horizontal for E_0^q/E_1^q and vertical for $E_x^{q^*}$) means also a *local stability* in sense of Lyapunov (neutral but *not* asymptotic stability), although the only points being attracted to each particular fixed point belong to its stable set having a zero Lebesgue measure in K. Recall that by definition a fixed point $(\tilde{x}, \tilde{q})$ is said to be locally stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any (x_0, q_0) being $\|(x_0, q_0) - (\tilde{x}, \tilde{q})\| < \delta$ there holds $\|(x_t, q_t) - (\tilde{x}, \tilde{q})\| < \varepsilon$ for $t > 0$. In other words, if an initial condition is located sufficiently close to $(\tilde{x}, \tilde{q})$ (in a δ-neighborhood of it), then the respective orbit is also located close to $(\tilde{x}, \tilde{q})$ (inside its ε-neighborhood). Consider a fixed point $E(\tilde{x}, \tilde{q})$ that is a "good" (E_0^q), "bad" (E_1^q) or internal ($E_x^{q^*}$) equilibrium being stable along the respective direction (horizontal in the first two cases and vertical in the last one). In an arbitrarily small neighborhood $U(E)$, there is an uncountable number of other fixed points that are also stable in their respective direction. It means that if an initial condition is located close enough to E, the related orbit will approach one of the fixed points belonging to U. This corresponds exactly to the local neutral stability of E. Concerning the points $E_0^{q^*}$ and $E_1^{q^*}$, their stable sets confine the sets of initial conditions, orbits of which approach E_0^q, $q > q^*$, and E_1^q, $q < q^*$, respectively. It can be shown that the fixed point $E_{x_2}^{q^*}$ is locally stable, while the stable set of $E_{x_1}^{q^*}$ separates the set of initial conditions, orbits of which converge to $E_x^{q^*}$ with $x \in (0, x_1)$, from the set of initial conditions, orbits of which end up at $E_x^{q^*}$ with $x \in [x_2, 1)$.

[3] To be precise, if $\alpha\Delta_c = 1$, then $q^* = 0$ and $E_x^{q^*}$ compose the lower boundary of K. Nonetheless, these points are unstable and almost all orbits approach the points E_0^q, which is exactly the same behavior as for $q^* < 0$.

In Figs. 1 for two distinct values of α, we plot by different colors basins of attraction for fixed points of different types together with several typical orbits shown by orange lines. The light-blue/green color denotes the initial points, orbits of which belong to stable sets of E_0^q/E_1^q, while the violet and pink colors correspond to points, orbits of which eventually approach $E_x^{q^*}$. Blue dots mark stable fixed points.

For small α, that is high social stigma (Fig. 1(a)), the probability of dishonesty concurring is rather large. The part of points (with sufficiently large q) located arbitrarily close to the right boundary of K is included in the basin of E_0^q. Hence, even if the initial fraction of dishonest firms is close to one, the State can eliminate dishonest behavior, choosing a sufficiently high level of monitoring (cf. the orbits marked by "1" and "2"). However, if the State does not put enough effort to fight dishonesty, the situation can become worse: even if the initial fraction of dishonest firms is low, eventually all firms can become dishonest (cf. the orbit marked by "5"). There is also a portion of orbits, starting at moderate values of q, that end up at one of the internal fixed points $E_x^{q^*}$ (cf. the orbits marked by "3" and "4"). As mentioned above, the basins of attraction of fixed points of different types are separated by the stable sets of $E_0^{q^*}$ and $E_1^{q^*}$. Another observation is that for initial points above/below the line $q = q^*$, the fraction of dishonest firms decreases/increases along the respective orbit. Therefore, in the case in which the inner honesty of the economy is high, even if the initial fraction of dishonest firms is large, it is enough to set the monitoring level greater than q^*, in order to improve the situation with non-compliant behavior.

For large α, that is low social stigma (Fig. 1(b)), if the initial fraction of dishonest firms is large, then it is not possible to eliminate non-compliant behavior completely (the basin of E_0^q is isolated from the left boundary of K). In general, the statements concerning blue and green basins are as in the previous case of small α (cf. the orbits marked by "1" and "5"). The basin of attraction of the points $E_x^{q^*}$ is divided into two parts, separated by the stable set of $E_{x_1}^{q^*}$: the violet/pink part corresponds to the initial conditions, orbits of which end up at $E_x^{q^*}$ with $x \in (0, x_1)/x \in [x_2, 1)$, respectively. The former are associated with a smaller level of non-compliant behavior, and hence, are more desirable to reach. Moreover, differently from the case with high social stigma, not all

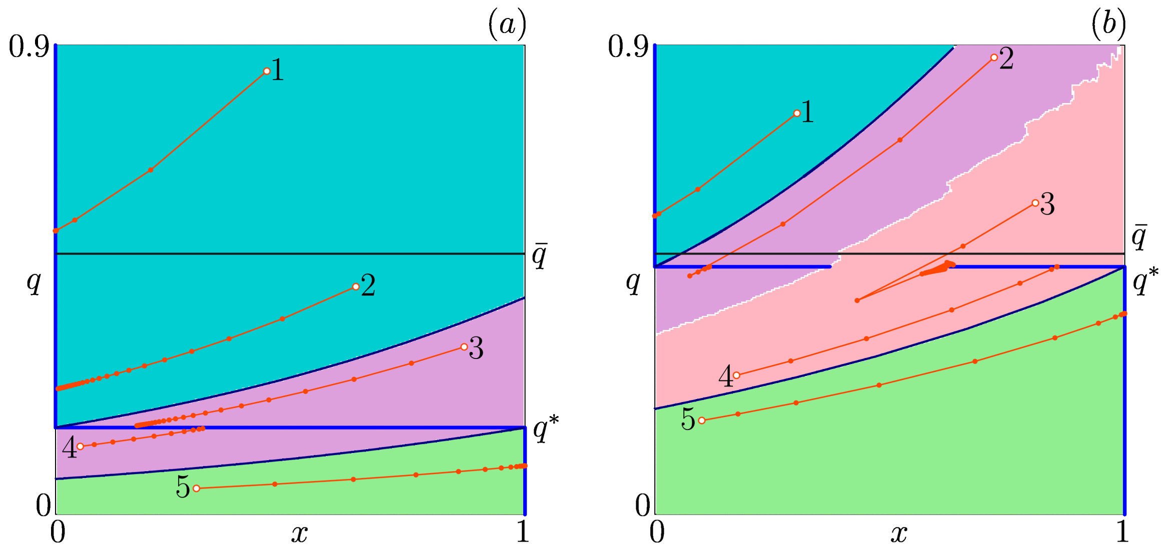

Fig. 1 Phase square K with different colors showing basins of fixed points of different types. Initial conditions from the light-blue/green region are attracted to points E_0^q/E_1^q, while initial conditions from the violet/pink region are attracted to $E_x^{q^*}$ with $x \in (0, x_1)/x \in [x_2, 1)$. Several typical orbits are shown by orange lines and blue dots mark stable fixed points. Parameter values are $\Delta_c = 1, f = 2, \gamma = 0.9, d = 0.9$ and $\alpha = 1.5$ (a); $\alpha = 20$ (b).

orbits are monotone. When an orbit approaches a fixed point $E_x^{q^*}$, it starts oscillating (it happens because the eigenvalue λ_2 related to the vertical direction becomes negative). Due to this reason, even if after the first several iterations the fraction of dishonest firms becomes essentially lower in comparison with the initial value, the situation can become worse after a while and will not improve afterwards (cf. the orbit marked by "3"). Certainly, the inherent integrity of a nation serves as a factor that a government can influence, but primarily over the medium to long term. By examining and contrasting countries with varying levels of integrity, we can assess to what extent the effectiveness of economic policies in combating dishonesty is influenced by initial conditions and the predisposition of the specific country in question. In fact, in the case of an irresponsible society, only a sufficiently large level of monitoring can essentially decrease non-compliant behavior. Our analysis confirms the results of [4]: the culture of legality, reflecting the intrinsic honesty of a country, is both necessary and sufficient to combat corruption. Only countries with a high degree of honesty can effectively eradicate dishonest practices. In contrast, in countries where the culture of honesty is lacking, minimal penalties for detected dishonesty can lead the economy to perpetually gravitate towards a state of entrenched illegality.

Notably, as further numerical experiments show, for a fixed α, a fixed d and an increasing f, the basins of "bad" and internal equilibria squeeze, while the basin of "good" equilibria enlarges. Such a dependence is expectable, since with increasing the fine size, being dishonest is related to higher risks. In case of a fixed f and an increasing d, the basins of boundary equilibria (both, "good" and "bad") squeeze, while the basin of internal equilibria enlarges. Understanding this latter dependence is left for future investigation.

We conclude this section by observing that for the map S given in (16) with using equation (12) for updating the monitoring level, asymptotic dynamics is represented by infinitely many fixed points of different types (with different levels of dishonesty ranging from 0 to 1), while more complex solutions are not possible. It is natural to wonder whether a particular choice of the function for updating q_t can be crucial for the model. In what follows we show that introducing even only slight changes to the function G has an essential influence on the overall dynamics of the map.

5 Updating based on relative variation

In this section we introduce a slightly modified expression for X_t, which can also be considered as a natural intuitive way to define it. Namely, we replace the absolute growth of the fraction of dishonest firms given by (12), with its relative growth as follows

$$X_t = \frac{x_{t+1} - x_t}{x_t}. \tag{30}$$

It is easy to show that also in this case $X_t \in [-1, 1]$. Then functions for updating the monitoring level become:

$$G_1(x_t, q_t) = \min\left\{ q_t \left(1 + d(1 - x_t) \frac{\alpha(\Delta_c - f q_t) - 1}{\alpha(\Delta c - f q_t) + 1} \right), \gamma \right\} \tag{31}$$

and

$$G_2(x_t, q_t) = \min\left\{ q_t \left(1 + d(x_t - 1)\right), \gamma \right\}. \tag{32}$$

We denote the updated map as $\tilde{S}$ to avoid confusion. It can be shown in a straightforward way that q_{t+1} obtained from either (31) or (32) belongs to the interval $[0, \gamma]$. Consequently, the region of definition for the map $\tilde{S}$ remains the same, that is, $\tilde{S} : K \to K$.

5.1 Simple dynamics

The map $\tilde{S}$ with G_1 and G_2 given in (31) and (32), respectively, still has an uncountable number of fixed points. The distinction from the map S with X_t being the absolute difference of x_{t+1} and x_t, is that now there are only two "good" equilibria $E_0^0(0,0)$ and $E_0^{q^*}(0,q^*)$. The point E_0^0 is asymptotically stable if $\alpha\Delta_c \leq 1$ and unstable otherwise. However, there is still continuum of "bad" equilibria $E_1^q(1,q)$, $q \in [0,\gamma]$, as well as continuum of internal equilibria $E_x^{q^*}(x,q^*)$, $x \in (0,1)$ and q^* as in (17).

Proposition 5.1. *The fixed point E_0^0 of the map $\tilde{S}$ is asymptotically stable if $\alpha\Delta_c \leq 1$ and is unstable otherwise. To a fixed point E_1^q, $q \in [0,\gamma]$, conditions (B.i)–(B.ii)) apply. For fixed points $E_x^{q^*}$, $x \in [0,1)$, there hold:*

(D.i) For $\alpha\Delta_c \leq 1$, all fixed points $E_x^{q^*}$ are located outside the definition square K.[4]

(D.ii) For $\alpha\Delta_c > 1$ and $(\alpha\Delta_c - 1)d \leq 4$, all points $E_x^{q^*}$ are locally stable.

(D.iii) For $(\alpha\Delta_c - 1)d > 4$, the points $E_x^{q^*}$ with $x < \hat{x}$ are unstable and with $x > \hat{x}$ are locally stable. For the latter their cumulative basin of attraction is confined by the stable set of $E_{\hat{x}}^{q^*}$.

Proof. The eigenvalues of the Jacobian matrix of E_0^0 are

$$\lambda_1 = \frac{2\Delta_c\alpha}{\Delta_c\alpha + 1} \quad \text{and} \quad \lambda_2 = 1 + \frac{d(\Delta_c\alpha - 1)}{\Delta_c\alpha + 1}.$$

They are both located inside the unit circle if $\Delta_c\alpha < 1$ and outside it if $\Delta_c\alpha > 1$. For the case $\Delta_c\alpha = 1$, when both $\lambda_i = 1$, $i = 1,2$, asymptotic stability can be shown by considering an orbit of a point in the neighborhood of E_0^0.

The Jacobian matrices of E_1^q are given by (22) and (23), and hence, the conclusions about their stability are as before (the same as for the map S).

[4] Again, if $\alpha\Delta_c = 1$, the points $E_x^{q^*}$ compose the lower boundary of K, but asymptotic dynamics of $\tilde{S}$ is exactly the same as for $q^* < 0$.

As for the points $E_x^{q^*}(x, q^*)$, $x \in [0,1)$, the respective Jacobian becomes

$$J^* = \begin{pmatrix} 1 & -\dfrac{\alpha f x(1-x)}{2} \\ 0 & 1 - \dfrac{(\alpha\Delta_c - 1)d(1-x)}{2} \end{pmatrix}. \tag{33}$$

As before the eigenvalue $\lambda_1 = 1$ is related to the horizontal eigenvector $v_1(1,0)$, along which the point is neutrally stable. The second eigenvalue is

$$\lambda_2 = 1 - \frac{(\alpha\Delta_c - 1)d(1-x)}{2}, \tag{34}$$

related to the vertical eigenvector $v_2(0,1)$. There holds $\lambda_2 < 1$ for $\Delta_c\alpha > 1$ and $\lambda_2 > -1$ if

$$x > \hat{x} = 1 - \frac{4}{d(\alpha\Delta_c - 1)}. \tag{35}$$

Note that if $0 < (\alpha\Delta_c - 1)d \leq 4$, then $\hat{x} < 0$. □

In Fig. 2 we show, for two distinct values of α, basins of attraction for various attractors together with several typical orbits. For small α (high social stigma, Fig. 2(a)), a sufficiently intensive monitoring is needed to decrease the number of dishonest firms. And in contrast to the map S from the previous section (with the update monitoring function depending on the absolute difference $x_{t+1} - x_t$), for the map $\tilde{S}$ with small α the non-compliant behavior, in general, cannot be eliminated completely. The State can only decrease the number of dishonest firms putting in enough effort. Therefore, when the inner honesty of society is high, using a monitoring technology that leads to the map S is more preferable than using a technology implying the map $\tilde{S}$, because in the former case the State can reach better results in fighting dishonesty.

The situation changes, however, for larger values of α (Fig. 2(b)). The first observation is that with increasing α the fixed point $E_{\hat{x}}^{q^*}$ moves towards the right border of K. The stable set of $E_{\hat{x}}^{q^*}$ confines the region of initial conditions producing orbits with trivial asymptotic dynamics. Therefore, for larger α this region naturally shrinks. The second observation, which is surprising, is that a nontrivial topological attractor with a significant basin can appear at the left border of the definition square K (that is, for $x = 0$).

In the remainder of the paper, we describe certain properties of these attractors, as well as examining their evolution with increasing α. In particular, we show that a cycle of such kind (located on the left border of K) may lead to appearance of another attracting cycle of the same periodicity, but with a single point on the upper boundary of K and all other points belonging to the interior of K. As a consequence, for certain parameter ranges, coexistence of several nontrivial attractors is observed, sometimes even with a complex structure of their basins.

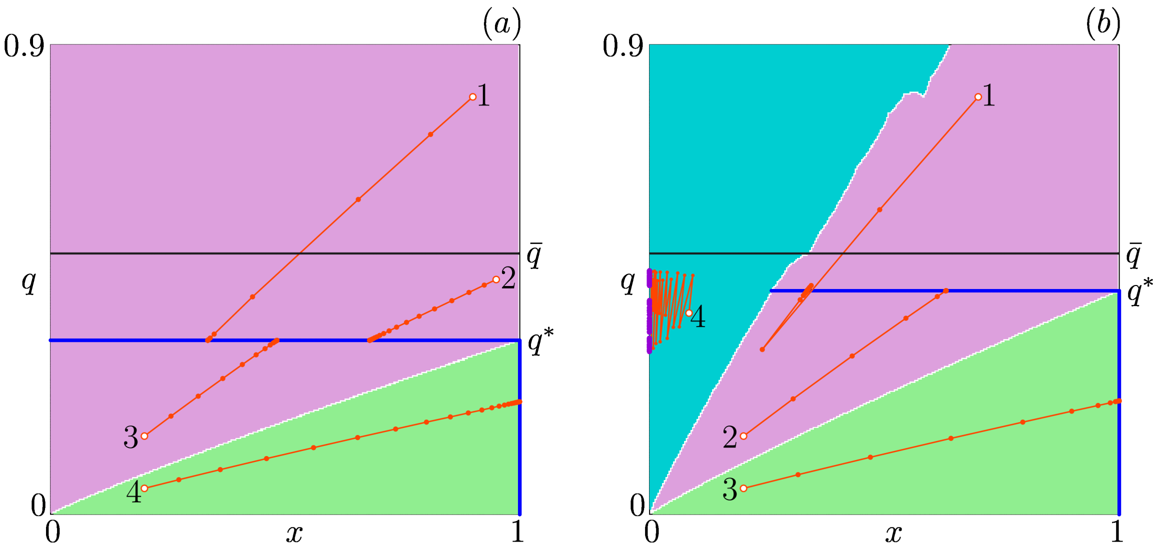

Fig. 2 Phase square K with different colors showing basins of different attractors of $\tilde{S}$. Initial conditions from the green region are attracted to points E_1^q, while initial conditions from the violet region are attracted to $E_x^{q^*}$ with $x > \hat{x}$. Orbits having initial conditions belonging to the light-blue region eventually end up at the vertical line $x = 0$ (dark-violet dots). Several typical orbits are shown by orange lines and blue dots mark stable fixed points. Parameter values are $\Delta_c = 1, f = 2, \gamma = 0.9, d = 0.9$ and $\alpha = 3$ (a); $\alpha = 7$ (b).

5.2 Nontrivial dynamics

Before studying the properties of nontrivial solutions of $\tilde{S}$ we notice that the line

$$L_0 = \{(0, q) \,:\, 0 \leq q \leq \gamma\}$$

is invariant with respect to $\tilde{S}$. Let us consider the restriction of $\tilde{S}$ to L_0, which is given by a 1D piecewise smooth map $\psi : [0, \gamma] \ni q_t \to q_{t+1} \in [0, \gamma]$ such that

$$q_{t+1} = \psi(q_t) = \begin{cases} \psi_1(q_t) = \min\left\{ q_t \left(1 + d\dfrac{\alpha(\Delta_c - f q_t) - 1}{\alpha(\Delta_c - f q_t) + 1}\right), \gamma \right\}, & \text{if} \quad q_t \leq \bar{q}, \\ \\ \psi_2(q_t) = q_t \left(1 - d\right), & \text{if} \quad q_t > \bar{q}. \end{cases} \tag{36}$$

The form of ψ influences the asymptotic dynamics along L_0 and, consequently, the dynamics of $\tilde{S}$, since the respective asymptotic solution belonging to L_0 can be topological or the Milnor attractor. The map ψ consists of the unimodal branch ψ_1 and the linear branch ψ_2. As $(1 - d) \in (0, 1)$, there is always $\psi(q) < q$ for $q \in [\bar{q}, \gamma]$. If there holds

$$\psi_1^M := \max_{q \in [0,\gamma]} q \left(1 + d\frac{\alpha(\Delta_c - f q) - 1}{\alpha(\Delta_c - f q) + 1}\right) = \frac{\left(\sqrt{(d+1)(\alpha\Delta_c + 1)} - \sqrt{2d}\right)^2}{\alpha f} \leq \gamma, \tag{37}$$

the unimodal branch ψ_1 is smooth. Otherwise it is nonsmooth having a flat top part $\psi_1(q) = \gamma$ for $q_- \leq q \leq q_+$ with $q_\pm$ obtained from

$$q + dq\frac{\alpha(\Delta_c - f q) - 1}{\alpha(\Delta_c - f q) + 1} = \gamma.$$

The asymptotic solution of ψ also depends on whether its absorbing interval spreads over the linear branch ψ_2 or not. Namely, another condition on which the asymptotic dynamics of $\tilde{S}$ may depend is

$$\psi^M = \max_{q \in [0,\gamma]} \psi(q) = \min\{\psi_1^M, \gamma\} = \bar{q}. \tag{38}$$

Describing in detail asymptotic dynamics of ψ and determining its influence on solutions of $\tilde{S}$ requires much deeper analysis. This task lies beyond the scopes of the current paper, although several preliminary remarks in this respect are given below, in order to offer suggestions for possible politics of the State concerning the monitoring principles.

While changing the honesty propensity that characterizes a society is a difficult goal to achieve, at least in the short term, policy instruments can be combined to combat dishonest behavior related to the level of fines (i. e., the punishment for a company found to be dishonest can be increased) and the ability to change the level of supervision in response to a change in the relative level of dishonesty (i. e., rapidly increasing resources to combat non-compliant behavior). These instruments are measured by the constants f and d and the following study aims at describing the joint role of such policy instruments.

In Fig. 3 in the (d, f) parameter plane, we show several regions separated by curves, obtained from (37) and (38) with using the equality sign. In each of these regions the map ψ attains qualitatively different forms, which influence asymptotic dynamics of the 2D map $\tilde{S}$. For instance, in the region marked by "I" (orange) and "II" (light-gray), attractors of ψ are defined by the smooth unimodal branch ψ_1. Even if in the region "II" there is $\gamma > \bar{q}$, the absorbing interval does not include the border (kink) point, and asymptotic dynamics of ψ is described by the same principles as in the region "I". On the contrary, in the region marked by "IV" (pink) there is $\psi_1^M > \gamma$ and $\gamma > \bar{q}$. Therefore, the map ψ is nonsmooth having three kink points: $q = q_-$, $q = q_+$, and $q = \bar{q}$, all located inside the absorbing interval.

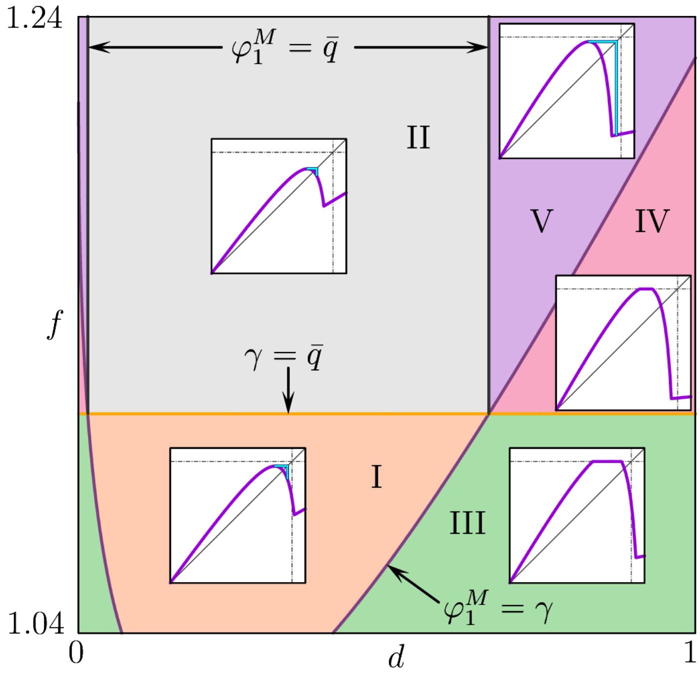

Fig. 3 Regions in the (d, f) parameter plane, in which the 1D map ψ attains qualitatively different forms. Parameter values are $\Delta_c = 1, \alpha = 11, \gamma = 0.9$.

In Fig. 4, we plot a 2D bifurcation diagram for the map $\tilde{S}$ in the (d, f) parameter plane for $\alpha = 11$ (rather low social stigma). The panel (b) is the magnification of the part of the diagram shown in the panel (a). The black lines denote the same bifurcation borders as in Fig. 3 given by (37) and (38) with using the equality sign. One can immediately notice the difference of the bifurcation structures inside regions I–V corresponding to qualitatively different forms of ψ.

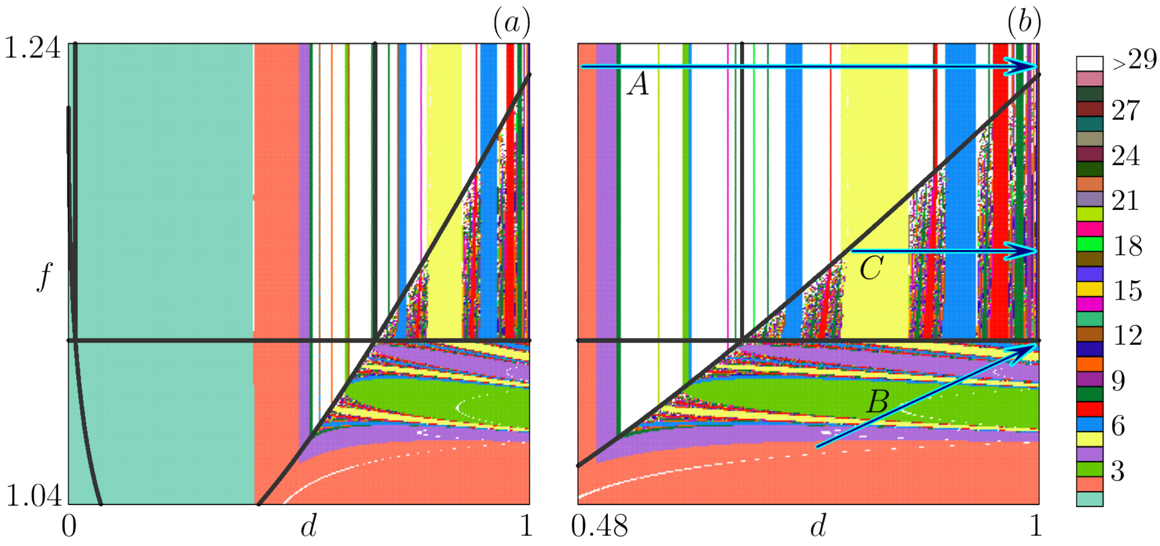

Fig. 4 The 2D bifurcation diagram in the (d, f) parameter plane of the map $\tilde{S}$. The other parameters are $\Delta_c = 1, \gamma = 0.9, \alpha = 11$.

A smooth unimodal branch (regions I, II, and V).

Let us consider the dynamics of $\tilde{S}$ inside the regions I, II, and V. As mentioned above, in the regions I and II the asymptotic dynamics of ψ is completely defined by the smooth unimodal branch ψ_1, while in the region V the linear branch ψ_2 also counts. In Figs. 5(a),(b) we plot a 1D bifurcation diagram for $\tilde{S}$ along the arrow marked by "A" in Fig. 4(b). As expected, for $d < d_+$ (inside the region II, and the same situation is also observed inside the region I) there exists at least one[5] nontrivial attractor belonging to L_0 (i. e., at $x = 0$), while the asymptotic dynamics of q mimics the universal bifurcation structure for a unimodal map, known as a Sharkovsky ordering [17] or a box-within-a-box structure [18]. In Fig. 5(c) the state space is shown for $d = 0.59$ with the light-blue color filling

[5] In our numerical experiments for parameter values belonging to the regions I and II, we have not observed any other nontrivial attractors, except for the one belonging to L_0. However, we cannot guarantee that for some other values of α another internal attractor does not appear, besides the fixed points $E_x^{q^*}$ described in Sec. 5.1.

the basin of fixed points $E_x^{q^*}$ and E_1^q and the white color corresponding to the basin of the attractor belonging to L_0 (shown by orange dots).

For $\psi_1^M > \bar{q}$ (inside the region V), the absorbing interval of ψ includes the border point $q = \bar{q}$. The respective bifurcation structure is based on the combination of smooth bifurcations (from the nonlinear unimodal branch) and nonsmooth border collision bifurcations (BCBs). For instance, for moderate values of d (around 0.72), there exists the region P_6^1 related to the period six, for which both its boundaries are associated with smooth bifurcations, a fold for the right-hand side boundary and a flip for the left-hand side. On the contrary, for the region P_5 corresponding to the period five, both its boundaries are associated with BCBs. Another example is the region P_6^2 corresponding to another 6-cycle (for d around 0.9), for which its left-hand side boundary is related to the (smooth) flip bifurcation, while its right-hand side boundary is associated with a BCB. Moreover, for values of $d \gtrapprox 0.9$ one can observe regions corresponding to periods 5, 6, 7, 8 and so on. Such an ordering, the so-called *skew tent map* structure [11], is typical for 1D continuous piecewise monotone maps having two branches.

The final remark, concerning nontrivial asymptotic dynamics of $\tilde{S}$ for the parameter values belonging to the regions I, II, and V, is about coexistence. As shown by numerical experiments, for certain ranges of d, the map $\tilde{S}$ has two coexisting nontrivial attractors both belonging to L_0. See, e. g., the inset in Fig. 5(b), showing the magnification of the region outlined red in (a) with two different orbits plotted red and blue, respectively. Further in the panel (d) one can observe three different basins of attraction: the light-blue one related to fixed points, the yellow one related to the 5-cycle belonging to L_0 (black dots), and the white one associated with some other attractor also belonging to L_0 (orange dots). The basins of the two nontrivial attractors are riddled inducing uncertainty in the respective area of initial conditions.

A unimodal branch with the flat top (regions III and IV).

Now we turn to parameter values belonging to the region III, in which asymptotic dynamics is defined by the unimodal branch ψ_1 having the flat top $\psi_1(q) = \gamma$ for $q_- \leq q \leq q_+$ with $\gamma < \bar{q}$.

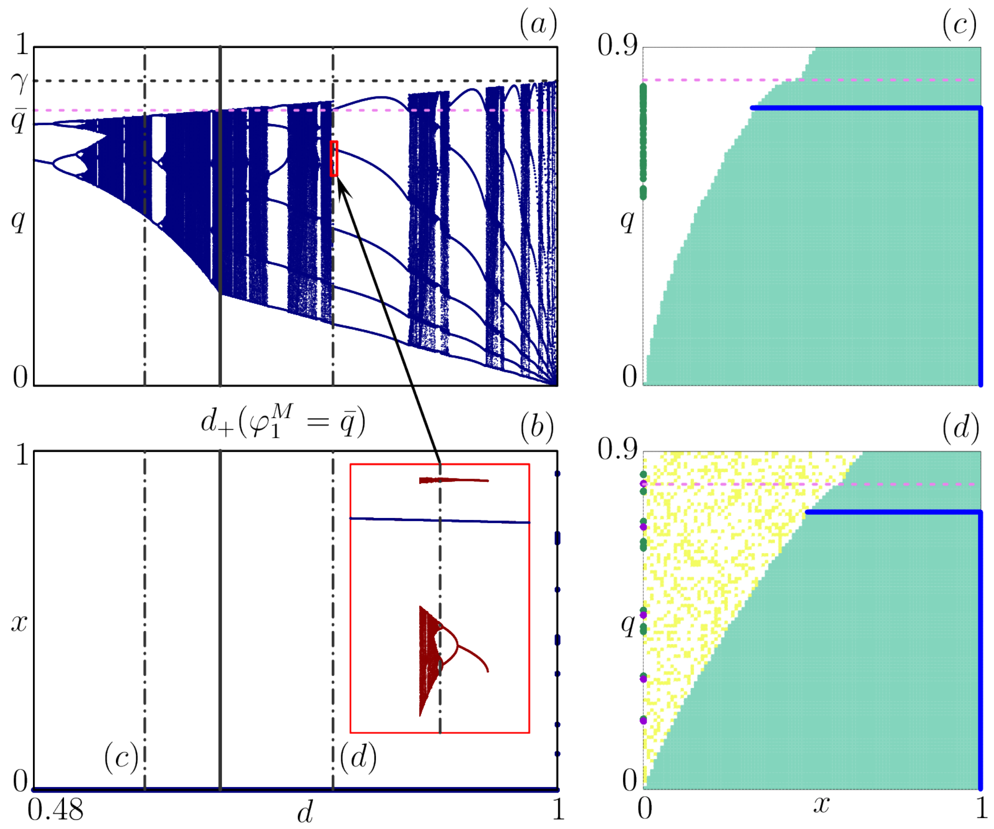

Fig. 5 (a), (b) The 1D bifurcation diagram versus d for the map $\tilde{S}$ with $\Delta_c = 1, \gamma = 0.9, \alpha = 11, f = 1.23$. (c), (d) The state plane (x, q) with basins of attraction related to different attractors of $\tilde{S}$. The basin of fixed points $E_x^{q^*}$ and E_1^q (blue dots) is colored light-blue; the basin of another attractor at L_0 (green dots) is colored white; in (d) the basin of the 5-cycle at L_0 (violet dots) is colored yellow. The bifurcation parameter is $d = 0.59$ (c) and $d = 0.777$ (d). The dashed magenta line is $q = \bar{q}$.

In Figs. 6(a), (b), we plot the 1D bifurcation diagram along the arrow marked "B" in Fig. 4(b). The orbits obtained by using different initial conditions are plotted with blue, red, and green colors, revealing for some parameter values, coexistence of at least two nontrivial attractors, one completely located on the line L_0 and another one located mostly in the interior of the definition square K (except for a single point with $q = \gamma$). See, for example, Fig. 6(c) for $d = 0.83, f = 1.08$, where the basins of the 3-cycle on L_0 and the 5-cycle, having four points in the interior of K and a single point on its upper boundary, are shown by green and yellow colors, respectively. In Fig. 6(d) for $d = 0.86, f = 1.085$, we observe only one nontrivial attractor, namely, the 3-cycle with two points located in the interior of K and one point with $q = \gamma$.

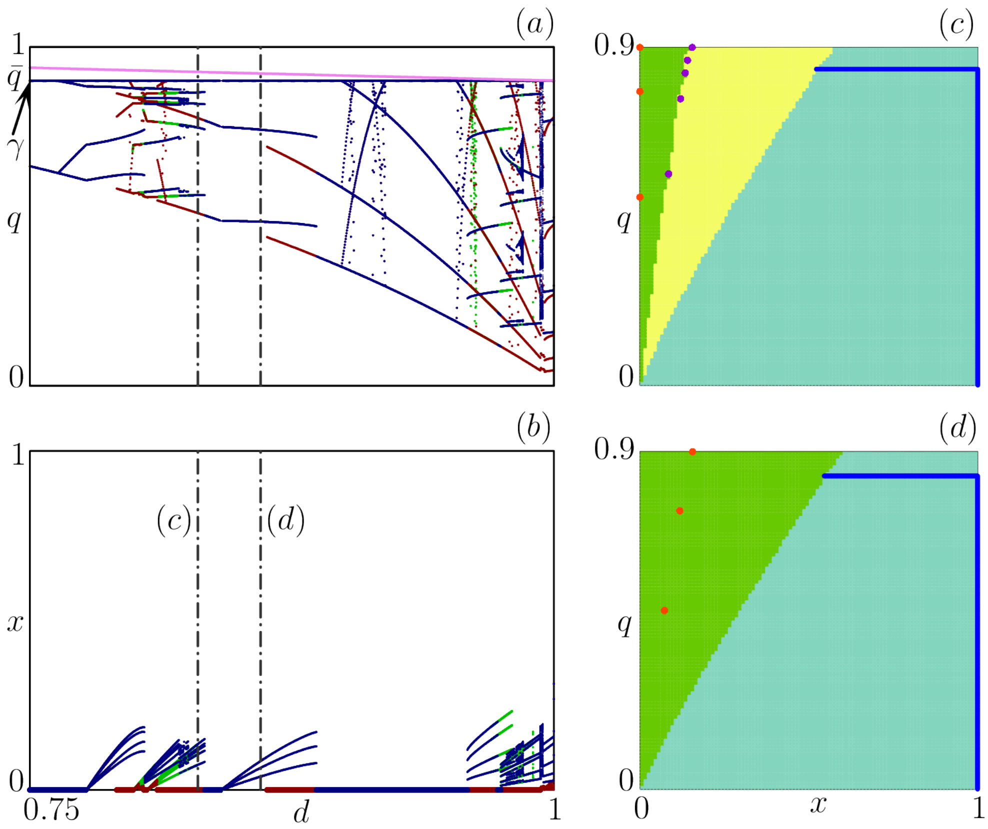

Fig. 6 (a), (b) The 1D bifurcation diagram versus d for the map $\tilde{S}$ with $\Delta_c = 1, \gamma = 0.9, \alpha = 11, f \in [1.065, 1.1], d \in [0.75, 1]$. (c), (d) The state plane (x, q) with basins of attraction related to different attractors of $\tilde{S}$. The basin of fixed points $E_x^{q^*}$ and E_1^q (blue dots) is colored light-blue; the basin of the 3-cycle (orange dots) is colored green; in (c) the basin of the 5-cycle (violet dots) is colored yellow. The bifurcation parameters are $f = 1.08, d = 0.837$ (c) and $f = 1.0884, d = 0.88$ (d).

The mechanism of occurrence of such "almost interior" attractors seems to be as follows. At first, an attracting n-cycle for some n occurs at L_0. Then it becomes unstable transversely (in the horizontal direction) and there occurs a new n-cycle with a single point at the upper boundary of K and all other points in the interior of K. At the bifurcation moment the two n-cycles coincide and afterward the cycle on L_0 becomes unstable. By changing further the bifurcation parameter (with increasing d), an interior n-cycle moves to the right. At another bifurcation value a new attracting m-cycle occurs at L_0, and for a certain range of parameters two attracting cycles, of periods n and m, coexist. Note that for values of d close to one, bifurcations become more frequent, and there decreases the parameter distance between the occurrence of different cycles on L_0 with their consecutive stability loss (giving birth to "almost interior" cycles of the same periodicity). As a consequence, for larger d one can observe more than two coexisting cycles of different periods, one of them located on L_0 and the others being "almost interior", having a single point with $q = \gamma$. The basins of some of these cycles can be riddled. Further investigation of the underlying mechanisms we leave for future works.

Finally, we consider the parameter region IV, in which the 1D restriction ψ has three border points, i. e., $q = q_-$, $q = q_+$, and $q = \bar{q}$. In Figs. 7(a) and (b) we plot a 1D bifurcation diagram along the arrow marked "C" in Fig. 4(b). According to numerical experiments, for most parameter values, besides the attracting fixed points $E_x^{q^*}$ and E_1^q, there exists a single nontrivial attractor located on L_0. As shown, for example, in Fig. 7(c) for $d = 0.91$, where the basins of attraction for the fixed points (blue dots) and the attracting 6-cycle (orange dots) are colored light-blue and blue, respectively. Coexistence of several nontrivial attractors is attained for values of d close to one. In Fig. 7(d) for $d = 0.996$, besides the basin (colored light-blue) associated with the fixed points, the basins of the 5-cycle (dark-violet dots), the 6-cycle (black dots), the 7-cycle (green dots), and the 11-cycle (orange dots) filled with yellow, blue, red and dark-blue, respectively are plotted. The 5-, 6-, and 7-cycles are "almost interior" with a single point on the upper boundary of the definition square K and all other points in the interior of K. The 11-cycle is located on L_0.

In general, concerning asymptotic dynamics of the map $\tilde{S}$ using the relative growth of the dishonest firms fraction, the following can be summarized. For all fine f levels, if d is not too large, i. e., the monitoring level q_t is not too sensitive with respect to the relative variation X_t (30) of the number x_t of dishonest firms, the system converges to a fixed point, describing a society in which honest and dishonest firms coexist (for the parameter values as in Fig. 4, this is related to $d < 0.4$). On the other hand, for larger values of d, more complex situations, when several nontrivial attractors coexist, may occur. Due to such coexistence and complexity of the related basins, choosing a fine level to reduce dishonest behavior becomes a difficult task. If the basins of some attractors are riddled, it is difficult to give the policy suggestions. In this respect, an interesting direction of further

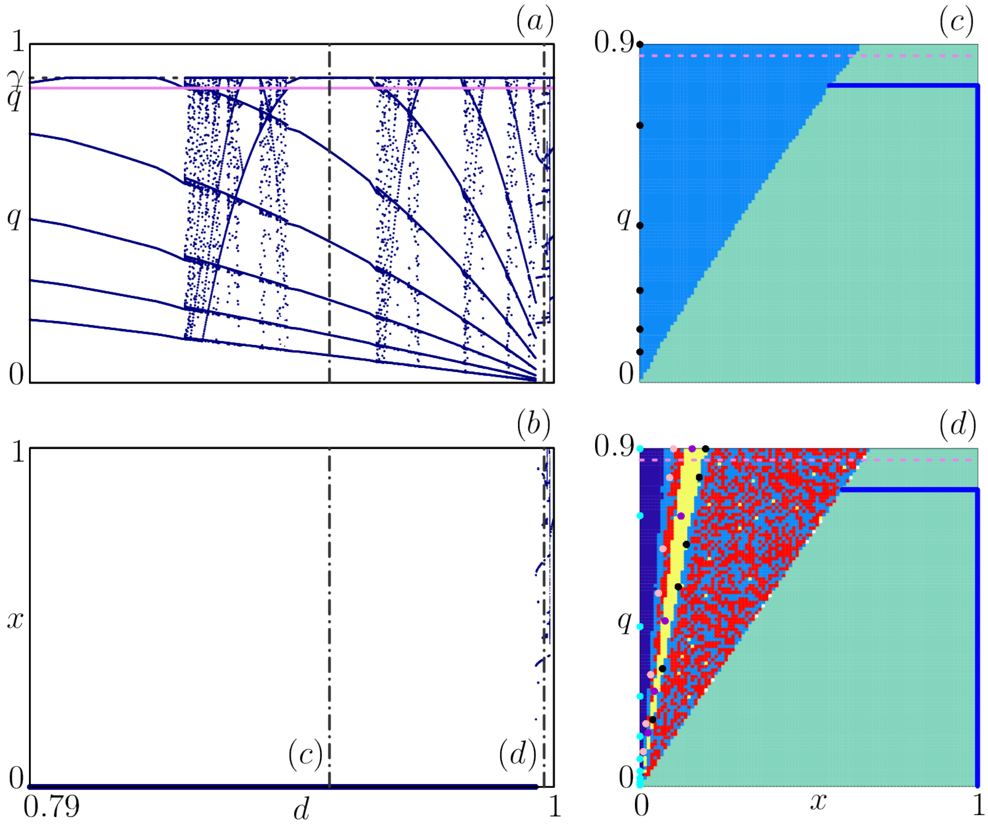

Fig. 7 (a), (b) 2D bifurcation diagram for $\Delta_c = 1, \gamma = 0.9, \alpha = 11, f = 1.15$. (c), (d) State plane (x, q) with basins of attraction related to different attractors of $\tilde{S}$ (the meaning of colors is as in Fig. 4). The basin of fixed points $E_x^{q^*}$ and E_1^q (blue dots) is colored light-blue. In (c) the basin of the 6-cycle at L_0 (black dots) is colored blue. In (d) the basin of the 11-cycle at L_0 (light-blue dots) is colored dark-blue; the basin of the 5-cycle (violet dots) is colored yellow; the basin of the 6-cycle (black dots) is colored blue; the basin of the 7-cycle (pink dots) is colored red. The bifurcation parameter is $d = 0.91$ (c) and $d = 0.996$ (d). The dashed magenta line is $q = \bar{q}$.

development could be considering a non-deterministic version of the model with added stochastic terms to one or both map components. However, these studies are left for future works.

6 Conclusion and further development

Public procurement acts as the mechanism by which governments secure the goods, services, and projects they need to function and provide services to their citizens. It is a vital tool for driving economic growth and encouraging market competition. Understanding the risks of corruption and illegal behaviors within public procurement is crucial, especially given the potential for collusion between public officials and private agents. This collaboration can lead to the misuse of public power for personal gain due to the high rewards for illegal and corrupt practices and low chances of detection. Therefore, maintaining transparency and establishing efficient organizational structures is essential for ensuring integrity in procurement processes.

In this context, we investigated how the level of public auditing influences the prevalence of illegal activities, particularly fraud, in the public procurement sector. This study aims at exploring whether the choice of function used to update monitoring levels significantly impacts the modeling of the overall dynamic process. With this aim in mind, we investigated a forward-looking updating control mechanism based on both the absolute and the relative variation of the expected dishonesty level and we observed that even minor changes introduced to the monitoring function can have a substantial effect on the asymptotic dynamics of the final model.

More precisely, the forward-looking mechanism for updating the State's monitoring level is based on the *absolute* growth of the dishonest firms fraction, qualitative dynamics are simple. From a policy perspective, the inner honesty plays a crucial role in the capability of the State to reduce non-compliant behavior, since in the case of an irresponsible society, only a sufficiently large level of monitoring can essentially reduce non-compliant behavior. However, strengthening the punishment (the fine) for dishonest behavior can increase the probability to concur frauds as well.

When, the monitoring level update mechanism is based on the *relative* variation of the dishonest firms fraction high inner honesty does not help, in general, to eliminate non-compliant behavior. Surprisingly though, in a society with a moderate level of endogenous dishonesty, provided that the initial fraction of dishonest firms is not too large, a sufficiently intensive monitoring can drive the system to a solution with all agents being honest. Hence, policy suggestions are difficult to be given and setting the fine level to reduce dishonest behavior becomes a challenging choice. Therefore, only countries with a high degree of honesty can effectively eradicate dishonest practices. In contrast, in countries where the culture of honesty is lacking, minimal penalties for detected dishonesty can lead the economy to perpetually gravitate towards a state of entrenched illegality.

Possible developments of our model could take into account the following points. First, the level of fines can be variable, e. g., it can be increased as the absolute or the relative variation of dishonest behavior increases. Second, we can consider that the firm does not know the future level of monitoring when calculating expected utility, i. e., some mechanism related to the formation of expectations about the probability of being detected could be added. Third, revenues (from fine) and costs (from audit) can be considered to design a more elaborate monitoring strategy for the government.

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Competing interests

The authors have no competing interests to declare that are relevant to the content of this article.

References

[1] Tanzi, V., Davoodi, H.: Corruption, growth, and public finances. In: Jain, A.K. (ed.) The Political Economy of Corruption, pp. 89–110. Routledge, London (2001). https://doi.org/10.4324/9780203468388

[2] Bajari, P., Tadelis, S.: Incentive versus transaction costs: a theory of procurement contracts. Rand J. Econ. **32**(3), 387–407 (2001) https://doi.org/10.2307/2696361

[3] Brianzoni, S., Coppier, R., , Michetti, E.: Evolutionary effects of non compliant behavior in public procurement. Struct. Chang. Econ. Dyn. **51**, 106–118 (2019) https://doi.org/10.1016/j.strueco.2019.08.008

[4] Coppier, R., Grassetti, F., Michetti, E.: Non-compliant behavior in public procurement: an evolutionary model with endogenous monitoring. Decis. Econ. Finance **44**(1), 459–483 (2021) https://doi.org/10.1007/s10203-021-00317-y

[5] Fisman, R., Miguel, E.: Corruption, norms, and legal enforcement: evidence from diplomatic parking tickets. J. Pol. Econ. **115**(6), 1020–1048 (2007) https://doi.org/10.1086/527495

[6] Zaklan, G., Westerhoff, F., Stauffer, D.: Analysing tax evasion dynamics via the ising model. Journal of Economic Interaction and Coordination **4**, 1–14 (2009)

[7] Nusse, H.E., Yorke, J.A.: Border-collision bifurcations including period two to period three for piecewise smooth systems. Phys. D **57**, 39–57 (1992) https://doi.org/10.1016/0167-2789(92)90087-4

[8] Banerjee, S., Karthik, M.S., Yuan, G., Yorke, J.A.: Bifurcations in one-dimensional piecewise smooth maps—theory and applications in switching circuits. IEEE Trans. Circ. Syst. I **47**, 389–394 (2000) https://doi.org/10.1109/81.841921

[9] Avrutin, V., Schanz, M., Gardini, L.: On a special type of border-collision bifurcations occurring at infinity. Phys. D **239**, 1083–1094 (2010) https://doi.org/10.1016/j.physd.2010.02.015

[10] Zhusubaliyev, Z.T., Mosekilde, E.: Bifurcations and Chaos in Piecewise-smooth Dynamical Systems. World Scientific, Singapore (2003). https://doi.org/10.1142/5313

[11] Avrutin, V., Gardini, L., Sushko, I., Tramontana, F.: Continuous and Discontinuous Piecewise-Smooth One-dimensional Maps: Invariant Sets and Bifurcation Structures. World Scientific, Singapore (2019). https://doi.org/10.1142/8285

[12] Panchuk, A., Sushko, I., Michetti, E., Coppier, R.: Revealing bifurcation mechanisms in a 2d nonsmooth map by means of the first return map. Commun. Nonlinear Sci. Numer. Simul. **117**, 106946 (2022) https://doi.org/10.1016/j.cnsns.2022.106946

[13] Bose, N., Capasso, S., Murshid, A.P.: Threshold effects of corruption: theory and evidence. World Dev. **36**(7), 1173–1191 (2008) https://doi.org/10.1016/j.worlddev.2007.06.022

[14] Banerjee, A., Fudenberg, D.: Word-of-mouth learning. Games Econ. Behav. **46**(1), 1–22 (2004) https://doi.org/10.1016/S0899-8256(03)00048-4

[15] Dawid, H.: On the dynamics of word of mouth learning with and without anticipations. Ann. Oper. Res. **89**, 273–295 (1999) https://doi.org/10.1023/A:1018983808923

[16] Lamantia, F., Pezzino, M.: Social norms and evolutionary tax compliance. Manch. Sch. **89**, 385–405 (2021) https://doi.org/10.1111/manc.12368

[17] Sharkovskii, A.N.: Problem of isomorphism of dynamical systems. In: Proc. 5th International Conference on Nonlinear Oscillations, vol. 2, pp. 541–544 (1969)

[18] Mira, C.: Chaotic Dynamics. World Scientific, Singapore (1987). https://doi.org/10.1142/0413