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ORIGINAL RESEARCH

An evolutive model of a boundedly rational consumer with changing preferences and reference group consumption

Gian Italo Bischi[1] · **Fabio Tramontana**[1]

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Abstract

In this paper, two dynamic models, recently proposed to describe the adaptive repeated choices of a boundedly rational consumer, are joined together. One considers a consumer adjusting the consumption level of a given good over time according to the observed discrepancy between expected and realized utility gain and modifies the utility function according to past consumption experience, also including saturation effects when past consumption is excessive. The other one considers the same adjustment mechanism with constant preferences but with a behavioral effect that introduces a tendency (or bias) to imitate a reference group of consumers. Merging these two models, a two-dimensional nonlinear dynamical system is obtained which describes consumers that decide their next period consumption of a given good by following two different (sometimes contrasting) criteria: their own utility maximization on the one side and imitation of a reference group of consumers on the other side. This leads to a greater uncertainty with respect to the model without the behavioral bias. Such uncertainty is studied through a numerical exploration of the long-run dynamics, guided by some global dynamical features of the nonlinear model, such as the folding action of the critical curves that characterize the behavior of the iterated noninvertible map and the singularities related to the presence of a vanishing denominator, namely focal points and prefocal curves. So, the aim of the paper is twofold: on the one side, it tries to contribute to the literature on the economic theory of boundedly rational consumers represented by evolutionary and behavioral approaches; On the other side, it tries to contribute to the recent literature about the global analysis of discrete dynamical systems characterized by contact bifurcations leading to the creation of complex topological structures of the attractors and their basins of attraction.

Keywords Consumer theory · Bounded rationality · Noninvertible maps · Focal points · Global bifurcations

✉ Fabio Tramontana
fabio.tramontana@uniurb.it

Gian Italo Bischi
gian.bischi@uniurb.it

[1] Department of Economics, Society, Politics (DESP), University of Urbino (Italy), Via Saffi 42, 61029 Urbino, Italy

1 Introduction

Consumers, in making their consumption choices, do not only consider their tastes, because they may also be influenced by a reference level of consumption see e.g. (Thaler, 1985; Kahneman et al., 1991; Tversky & Kahneman, 1991). For example, people belonging to a reference group (colleagues, members of the family, friends, and so on) that can be considered as a reference for the consumer who does not want to behave too differently from them, see e.g. (Kapteyn et al., 1980; Hayakawa, 2000; Janssen & Jager, 2001; Varela & Pritchard, 2011; Grohmann & Sakha, 2019). Moreover, by putting in relation current and future consumption, it is sometimes possible to observe that an increasing relation is found for low and moderate past consumption levels, whereas future consumption decreases if the quantity currently consumed is too high, which is a typical consumption saturation effect see (Barnett, 1973; Higgins, 1972; Varian, 2014). In other words, the consumption choice of a consumer can be influenced by preferences that can endogenously evolve according to the historical amount of goods consumed[1] and at the same time, the choice may not only reflect the preference but also the aspiration of a consumer to want to belong to a group, imitating the behavior of this reference group. The consequences of these effects on the consumption choices can be complicated and in this work we aim at shading some light on such an intriguing topic.

To do that we move from the basic framework developed in (D'Orlando & Rodano, 2006) by D'Orlando and Rodano, who proposed a dynamic model to describe adaptive consumers that update their consumption choices on the basis of the observed discrepancy between the expected utility gain from the consumption of a given good (measured by the price paid) and the effective utility gain. This gives rise to a one-dimensional dynamical system mimicking the behavior of boundedly rational consumers whose repeated choices follow a trial and error (or adaptive) method, because at each time they correct the previous choices on the basis of their observations. The unique stationary state in this dynamic process is a rational steady state, i.e. the same choice of a rational agent whose chosen quantity is given by the solution of the utility maximization problem (where the price equals the marginal utility). If such steady state is asymptotically stable under the adaptive process proposed, then an "evolutionary explanation" of the assumption of rational behavior is obtained. Indeed, as already noticed in (D'Orlando & Rodano, 2006), in this case one may say that the boundedly rational agents are able to learn from their past experience and become rational in the long run. In (Alchian, 1950) Alchian used a similar approach to show how economic agents (which were firms in that case) follow a "Darwinian" evolution.

However in (D'Orlando & Rodano, 2006) the model is further improved by assuming that also the parameter that characterizes the utility function is updated according to the consumption choice of the previous period, and the dynamic model becomes a discrete two-dimensional dynamic model, and whose time evolution is given by the iteration of a nonlinear map of the plane into itself, whose steady states represent local maximum points of the utility function, i.e. again the choices of a rational consumer. The long run dynamics of this two-dimensional model is first considered in (D'Orlando & Rodano, 2006) by assuming that the consumers increasingly prefer a good consumed in the past due to habits or skillness gained, i.e. the learning mechanism, connecting current to past consumption of the good considered, is based on an increasing function (more consumption now implies even more in the future).This means that the dynamic model is represented by the iteration of a nonlinear invertible two-dimensional map, whose steady states represent possible alternative choices

[1] Another possibility, not studied in this paper, is to consider goods whose consumption creates addiction for the consumer.

of rational consumers. The global dynamic properties of this model have been studied in (Bischi & Tramontana, 2007), where the presence of a denominator that can vanish is linked to the presence of certain singularities denoted as focal points and prefocal curves in (Bischi et al., 1999, 2003, 2005), whose presence strongly influences the structure of the basins of attraction.

A generalization of the model has recently been introduced in (Bischi & Tramontana, 2023), where the saturation effect has been considered. In that work this assumption leads to a unimodal (or one-hump) preference function instead of an increasing one, which implies that the two-dimensional map, whose iteration represents the time evolution of the consumer's choices, is transformed from an invertible map to a noninvertible one. This suggests that new global dynamic effects, related to the folding action of critical curves (a feature of noninvertible maps), are added to those due to the presence of focal points. These new global properties influence the structure of the attractors as well as their basins of attraction.

In the meantime, a modification to the one-dimensional model with constant preferences of D'Orlando and Rodano has been proposed in (Bischi & Tramontana, 2022): a behavioral component has been added to the demand side to take into account the results of some experiments conducted by behavioral economists, to show that the consumers, in making their consumption choices, can be influence by a reference level of consumption. In order to capture this effect, in (Bischi & Tramontana, 2022) a term has been introduced so that the consumption of the good is increased (resp. decreased) in the next period if currently it is lower (resp. higher) than the reference one. This implies that a steady state consumption, besides the consumer's own preferences, also reflects the consumption level of the reference group. As a consequence, as stressed in (Bischi & Tramontana, 2022), these steady states no longer correspond with rational steady states, as they also include the bias reflecting the influence of other consumers. So, even in the case of convergence to a steady state, the consumers do not learn to be rational, as they converge to a steady state quantity representing a compromise between their tastes and the consumption of the reference group. Moreover, if the behavioral bias is accentuated, then convergence to a steady state may fail, and consumption choices may vary erratically. At the same time, if the discrepancy between the consumption driven by tastes and the one driven by the others is quite large, convergence may be facilitated, thus confirming that the effects of behavioral features are not easily predictable.

The aim of this paper consists in combining, in a more general dynamic model, the two distinct effects discussed in (Bischi & Tramontana, 2022, 2023), namely the behavioral bias towards the reference group consumption and the saturation effect in the updating of the preference function.

As we shall see, these two combined effects will give rise to a more complicated analysis of the existence of steady states. Moreover, the parameter which measures the attitude of the consumers to imitate the choices of the reference group may have non monotonic effects, in the sense that in some ranges (namely for sufficiently low values) its increase enhances the stability of steady states, whereas in different ranges (for higher values) increasing values may lead to stability loss and the outcome of periodic or chaotic patterns due to overshooting phenomena, as is often observed in real systems. Numerical explorations of the model show more frequent situations of multistability, i.e. the coexistence of several attractors each with its own basin of attraction, so a problem of path dependence arises and the adaptive mechanism proposed becomes a device for the selection of the dynamic pattern prevailing in the long run. In such situations the role of initial conditions is crucial, and the delimitation of the basins of attraction must be considered in the study of the model. This requires a global analysis of the nonlinear model of the evolutive consumer proposed in this paper by using methods that are mathematically interesting. Indeed, effects of nonlinearity in economic modelling have been extensively studied in the literature. A common consequence of the coexistence of several attractors in nonlinear dynamic models is given by the property that small perturbations are recovered in as far as they are confined inside a neighborhood of a given attractor, whereas larger perturbations lead to time evolutions that further depart from the steady state and in the long run go towards a coexisting attractor, a situation that has been called "corridor stability" in (Leijonhufvud, 1973), see also (Dohtani et al., 2007). However, the results given in this paper also provide different situations when non-connected basins of attraction, formed by sets of disjoint portions of the basin, exist. In fact, the presence of non-connected basins can be described by saying that a small perturbation can be recovered by the endogenous dynamics of the evolutive model, a medium-size perturbation may lead to a different attractor, whereas an even larger perturbation may be recovered leading the system back to the original attractor. These situations can only be studied by global dynamic methods, which can usually be performed through heuristic methods obtained by a combination of analytical, geometrical and numerical approaches.

Even if the results of this paper are obtained for a particular dynamic model, the mathematical methods used to obtain these results are quite general, and the conclusions obtained about two kinds of complexity, related to complex attracting sets and complex structure of the basins of attraction, can be extended to general classes of discrete-time adaptive models. From a mathematical point of view, these methods involve the study of noninvertible maps as given in (Mira et al., 1996) as well as some global properties of maps with a vanishing denominator, see e.g. (Bischi et al., 1999; Bischi & Tramontana, 2007; Naimzada & Tramontana, 2009). In particular, the model studied in this paper gives us the opportunity to learn an important lesson, because in some parameters' ranges such that the steady state is locally stable, a global analysis shows that other attractors coexist with the locally asymptotically stable steady state, thus giving a strong path dependence. These dynamic scenarios clearly show the importance of a global analysis of nonlinear dynamical systems, because a study limited to local stability and bifurcations, based on the linear approximation of the model around the steady states, sometimes may be quite incomplete and even misleading.

To sum up, the aim of this paper is twofold: First we join two economic assumptions about the behavior of the evolutive boundedly rational consumer, namely the behavioral bias of imitating the consumption of a reference group and a saturation effect in the evolution of the preference function; Second we use mathematical methods for the global qualitative analysis of nonlinear dynamical systems, such as critical curves and focal points, to provide an exemplary study of the global bifurcations leading to the creation of complex topological structures of attractors and basins of attraction.

The paper is organized as follows. In Sect. 2, a description of the economic dynamic model is given. In Sect. 3, the existence of steady states is studied. In Sect. 4, some definitions related to the basins are given together with an analysis of global geometric properties of the dynamic model considered. In Sect. 5, these definitions and results are applied to the study of the global bifurcations of the basins of the economic dynamic model considered in this paper through some numerical simulations. Section 6 concludes and outlines further studies. In addition, a mathematical appendix is added to summarize some general properties of noninvertible maps and critical curves, in order to provide a more self-contained paper.

2 The model

Let x be the quantity of a given good and y the aggregated quantity of all the other goods that a consumer can buy. The utility $U(x, y)$ is a smooth function of its arguments that represents the satisfaction obtained by the consumer as a consequence of the consumption of the goods. If p is the unitary price of the good considered and the price of all other goods is conventionally taken as a reference unitary price, the budget constraint becomes

$$px + y = m \tag{1}$$

where m is the amount of money that a consumer can use to buy goods. A rational choice of the consumer is a solution (x^*, y^*) of the problem of maximization of U under the budget constraint (1). If we exclude corner solutions,[2] a rational solution is identified by

$$MRS = \frac{\partial U/\partial x}{\partial U/\partial y} = p;\ px + y = m \tag{2}$$

where MRS is the Marginal Rate of Substitution between the goods x and y, with the latter that can be interpreted as a composite good made up by all the other goods. In particular if $MRS = p$ the consumer has no incentive to change the goods composition, while if $MRS >$ $(<)$ p then there is the incentive to reallocate the budget m by increasing /decreasing) the amount of the good x. However, under the assumption of bounded rationality, in (D'Orlando & Rodano, 2006) the consumers are considered unable to compute the solutions of this problem, and they follow a time adjustment process:

$$x_{t+1} = x_t + \mu\left[MRS(x_t) - p\right] \tag{3}$$

where $\mu > 0$ represents the speed of adjustment. This adaptive process is based on the assumption that at any time period t the quantity x_{t+1} that the consumer decides to buy in the next period is obtained as a correction of the quantity chosen in the current period, x_t, according to the discrepancy between the given price p and the experienced relative utility gain $MRS(x_t)$. It is straightforward to notice that a steady state $x_{t+1} = x_t$ of this process is a rational choice, i.e. a solution of (2). Following (Bischi & Tramontana, 2022), we also consider a behavioral component by assuming that the consumer is influenced by the consumption level x_r of a reference group, and consequently we add a term in the dynamic equation such that the consumption of the good considered is increased (resp. decreased) in the next period if currently it is lower (resp. higher) than x_r

$$x_{t+1} = x_t + \mu\left[MRS(x_t) - p\right] + \gamma(x_r - x_t) \tag{4}$$

where the parameter $\gamma \geq 0$ measures how relevant is the influence of the reference consumption level x_r on the consumption choices of the given consumer.

In (D'Orlando & Rodano, 2006) a Cobb-Douglas utility function is considered:

$$U(x, y) = x^{\alpha} y^{1-\alpha} \tag{5}$$

where $x > 0$, $y > 0$ (i.e. $x < m/p$ according to (1)) and the real parameter $\alpha \in [0, 1]$ represents the amount of the good x needed to compensate a reduction of one unit of the composite good y (i.e. it parametrizes the steepness of the slope of the Cobb-Douglas utility function). Moreover, we can also say that parameter α measures the preference (marginal

[2] To exclude corner solutions it is sufficient to have asymptotic indifference curves and this is the case for the most used utility functions, such as Cobb-Douglas and CES utility functions.

utility) for good x (if we multiply the quantity x by a factor $k > 0$, then the utility is multiplied by a factor k^α). From (5) we get

$$MRS(x) = \frac{\alpha}{1-\alpha}\frac{y}{x} = \frac{\alpha}{1-\alpha}\frac{m-px}{x} \tag{6}$$

provided that $\alpha < 1$. In the following we assume $m > p$, so that small values of x are not associated with small y as well. Plugging (6) into (4) the adjustment process becomes

$$x_{t+1} = x_t + \frac{\mu}{1-\alpha}\left(\frac{m\alpha}{x_t} - p\right) + \gamma\left(x_r - x_t\right). \tag{7}$$

In (D'Orlando & Rodano, 2006) it is also assumed that the consumer's preferences may be influenced by past choices, i.e. the preference parameter is endogenized to become a dynamic variable depending on past consumption

$$\alpha_{t+1} = \alpha\left(x_t\right) \tag{8}$$

and the dynamic model becomes two-dimensional, the evolutive process being represented by the iteration of the two-dimensional map $(x_{t+1}, \alpha_{t+1}) = T\,(x_t, \alpha_t)$, where

$$T: \begin{cases} x_{t+1} = x_t + \frac{\mu}{1-\alpha_t}\left(\frac{m\alpha_t}{x_t} - p\right) + \gamma\left(x_r - x_t\right) \\ \alpha_{t+1} = \alpha\left(x_t\right) \end{cases} \tag{9}$$

The denominators in the first component of the map T are positive under the already stated conditions $x_t > 0$ and $0 \le \alpha_t < 1$. In (D'Orlando & Rodano, 2006) $\alpha\,(x)$ is assumed to be a continuous and increasing function, i.e. a consumer prefers to consume a good more and more according to the quantity consumed in the previous period, due to consumption habits or skillness gained because of past consumption, and they propose an increasing exponential sigmoidal function see also (Bischi & Tramontana, 2007). However, as the same authors suggest, one may assume that if past consumption is too high then the consumer becomes tired of that good, i.e. a saturation effect occurs related to a decreased necessity to buy that good. This suggestion has been taken up in (Bischi & Tramontana, 2023), where a continuous unimodal function is considered, similar to the one used in (D'Orlando & Rodano, 2006) for low values of consumption x whereas it becomes a decreasing function for high values of x. The particular functional form proposed in (Bischi & Tramontana, 2023) is

$$\alpha(x) = \frac{1}{k}x^2 e^{-hx} + l \tag{10}$$

with $kh^2 > \frac{4}{(1-l)e^2}$ in order to ensure $\alpha < 1$, the range of (10) being $\alpha \in \left(l, l + \frac{4}{4e^2kh^2}\right)$, as $\alpha(0) = l$ and it reaches its maximum value at $x = \frac{2}{h}$. The parameter $l > 0$ has been introduced to mimic, for low values of x, the shape of the increasing sigmoid function used in (D'Orlando & Rodano, 2006), whereas for $x > \frac{2}{h}$ (10) decreases and approaches the horizontal asymptote $\alpha = 0$ as $x \to +\infty$.

In the following we study the global dynamic behavior of the model (9) with (10).

3 Steady states

The steady states of the adaptive model described in the previous section are obtained by setting $x_{t+1} = x_t$ and $\alpha_{t+1} = \alpha_t$ in (9) and, considering the preference function (10), are given by the solutions of the system

$$\begin{cases} \frac{\mu}{1-\alpha}\left(\frac{m\alpha}{x} - p\right) + \gamma\left(x_r - x\right) = 0 \\ \alpha = \frac{x^2 e^{-hx}}{k} + l \end{cases}$$

whose solutions can be graphically represented as the intersections between the two curves of equation

$$\begin{cases} \alpha = \frac{\gamma x^2 - \gamma x_r x + \mu p x}{\gamma x^2 - \gamma x_r x + \mu m} \\ \alpha = \frac{x^2 e^{-hx}}{k} + l \end{cases} \tag{11}$$

In this paper we are mainly interested in the effects of the behavioral parameters x_r and γ, so we shall consider the second function, i.e. the unimodal preference function (10), as a fixed curve, and we study the modifications of the first one induced by variations of x_r and γ. First of all we stress that for $\gamma = 0$ it reduces to the straight line $\alpha = \frac{p}{m}x$, thus giving the rational steady states (from one to three) already studied in (Bischi & Tramontana, 2023). Even for $\gamma > 0$ the first function (11) starts from the origin, i.e. $\alpha(0) = 0$ but as $x \to +\infty$ it approaches the horizontal asymptote $\alpha = 1$ from above, i.e. $\alpha(+\infty) = 1^+$, and it intersects the line $\alpha = 1$ at $x = \frac{m}{p}$, i.e. $\alpha\left(\frac{m}{p}\right) = 1$. Moreover, if $\gamma x_r > \mu p$ then the curve intersects the horizontal axis $\alpha = 0$ at $x = x_r - \frac{\mu p}{\gamma}$. The denominator is always positive when $x_r < 2\sqrt{\frac{\mu m}{\gamma}}$, whereas for $x_r > 2\sqrt{\frac{\mu m}{\gamma}}$ it has two positive roots $0 < x_1 < x_2$ with $x_{1,2} = \frac{x_r}{2}\left(1 \pm \sqrt{1 - \frac{4\mu m}{\gamma x_r^2}}\right)$ at which the two vertical asymptotes $x = x_1$ and $x = x_2$ are located. Some situations are shown in Fig. 1, from which it can be seen that for $l > 0$ at least one steady state with positive consumption exists, and two further positive steady states consumption values can be obtained under different conditions on the parameters. The different number and coordinates of the steady states represented in the panels of Fig. 1 may have an interesting interpretation. From (Bischi & Tramontana, 2023) we know that the underlying model without behavioral features may be characterized by three steady states with increasing values of consumption and level of preference (we denote them here with E_l, E_m and E_h, respectively). The introduction of the tendency to imitate the behavior of a reference group may have the effect of solving this uncertain scenario in favour of one of the extreme steady states. For instance, panel c (resp. d) of Fig. 1 shows that if the reference consumption level is low (resp. high) enough the coexistence of steady states is solved and only the lowest (resp. highest) valued steady state remain. In other words, when the comparison with a reference group is introduced in the model, the steady state which is more distant from the reference consumption may be eliminated.

Unfortunately, an analytical computation of the steady states values cannot be obtained, and neither can the stability conditions for their creation/destruction via fold bifurcations related to tangency between the two curves. So, our study of the dynamical properties of the model considered in this paper will mainly rely on numerical explorations. However, some global dynamical properties of the nonlinear model (9) will guide our numerical analysis. Indeed, even if the nonlinear model (9) is represented by the iterated application of a non-invertible map with a denominator that can vanish, it allows the analytical computation of the equations of some curves, such as the critical curves see e.g. (Mira et al., 1996) and the prefocal curves see e.g. (Bischi et al., 1999), whose contacts with attractors or the boundaries of their basins of attraction give rise to some global bifurcation that may be useful to explain some global dynamic scenarios as well as their qualitative changes, i.e. some peculiar modifications of their topological structure.

The following section is dedicated to the analytic computation of these curves.

4 Global properties of the map

When a dynamical system has several coexisting attractors, a study of its global properties is necessary in order to understand the structure of the boundaries that separate the respective basins of attraction (see the Appendix for a summary of the main properties and definitions). Starting from the definition of stability, let U be a neighborhood of an attractor A whose points converge to A. Of course $U \subseteq \mathcal{B}(A)$, but also the points which are mapped inside U after a finite number of iterations belong to $\mathcal{B}(A)$. Hence, the *basin*of A is given by the open set $\mathcal{B}(A) = \bigcup_{n\geq 0} T^{-n}(U)$, where $T^0(x,\alpha) = (x,\alpha)$ and $T^{-n}(x,\alpha)$ represents the set of rank-n preimages of (x,α), i.e. the set of points that are mapped into (x,α) after n iterations of the map T. The basin $\mathcal{B}(A)$ is trapping under T and invariant under T^{-1}, i.e.

$$T^{-1}(\mathcal{B}(A)) = \mathcal{B}(A), \quad T(\mathcal{B}(A)) \subseteq \mathcal{B}(A)$$

The boundary behaves as a repelling set for the points near it, since it acts as a watershed for the trajectories of the map T. Points belonging to $\partial\mathcal{B}(A)$ are mapped into $\partial\mathcal{B}(A)$ both under forward and backward iteration of T. More exactly

$$T^{-1}(\partial\mathcal{B}(A)) = \partial\mathcal{B}(A), \quad T(\partial\mathcal{B}(A)) \subseteq \partial\mathcal{B}(A)$$

We remark that $T^{-1}(\partial\mathcal{B}(A)) = \partial\mathcal{B}(A)$ implies that if a curve segment belongs to $\partial\mathcal{B}(A)$ then also all its preimages must belong to $\partial\mathcal{B}(A)$. In particular, $\partial\mathcal{B}(A)$ includes the whole stable set of any fixed point (or cycle) of T belonging to $\partial\mathcal{B}(A)$. So, in order to study the structure of the boundaries of a basin, the properties of the inverse (or inverses if a map is noninvertible, see the Appendix, or (Mira et al., 1996) for more details) must be considered. The map T defined in (9) is noninvertible because given (x_{t+1}, α_{t+1}), with $\alpha_{t+1} \in \left(l, l + \frac{4}{e^2kh^2}\right)$, then two distinct preimages $0 < x_{t,1} \leq \frac{2}{h} \leq x_{t,2}$ are obtained from the second component, and for each $x_{t.i}$, $i = 1, 2$, we get the corresponding

$$\alpha_{t,i} = \frac{(x_{t+1} - x_{t.i})x_{t.i} + \mu p x_{t.i} - \gamma\,(x_r - x_{t.i})}{\mu m + (x_{t+1} - x_{t.i})x_{t.i}}.$$

Instead, if $\alpha_{t+1} = l + \frac{4}{e^2kh^2}$ then the two preimages merge into the same: $x_{t,1} = x_{t,2} = \frac{2}{h}$, and if $\alpha_{t+1} > l + \frac{4}{e^2kh^2}$ then no preimages exist. According to the number of preimages, this kind of noninvertible map is denoted as $Z_0 - Z_2$ map (see the Appendix, or (Mira et al., 1996) for more details).

4.1 Critical curves

As recalled in the Appendix, the global properties of a noninvertible map can be studied by using the method of critical curves LC (from the French " Ligne Critique") defined as the locus of points having two, or more, coincident rank-1 preimages, located in a set denoted by LC_{-1}. Analogously to the case of differentiable one-dimensional maps, where the derivative necessarily vanishes at the local extremum points, for a two-dimensional differentiable map LC_{-1} belongs to the set of points in which the Jacobian determinant vanishes, i.e. $LC_{-1} \subseteq \left\{(x,\alpha) \in \mathbb{R}^2 | \det J = 0\right\}$, and LC is obtained as the image of LC_{-1}, i.e., $LC = T(LC_{-1})$.

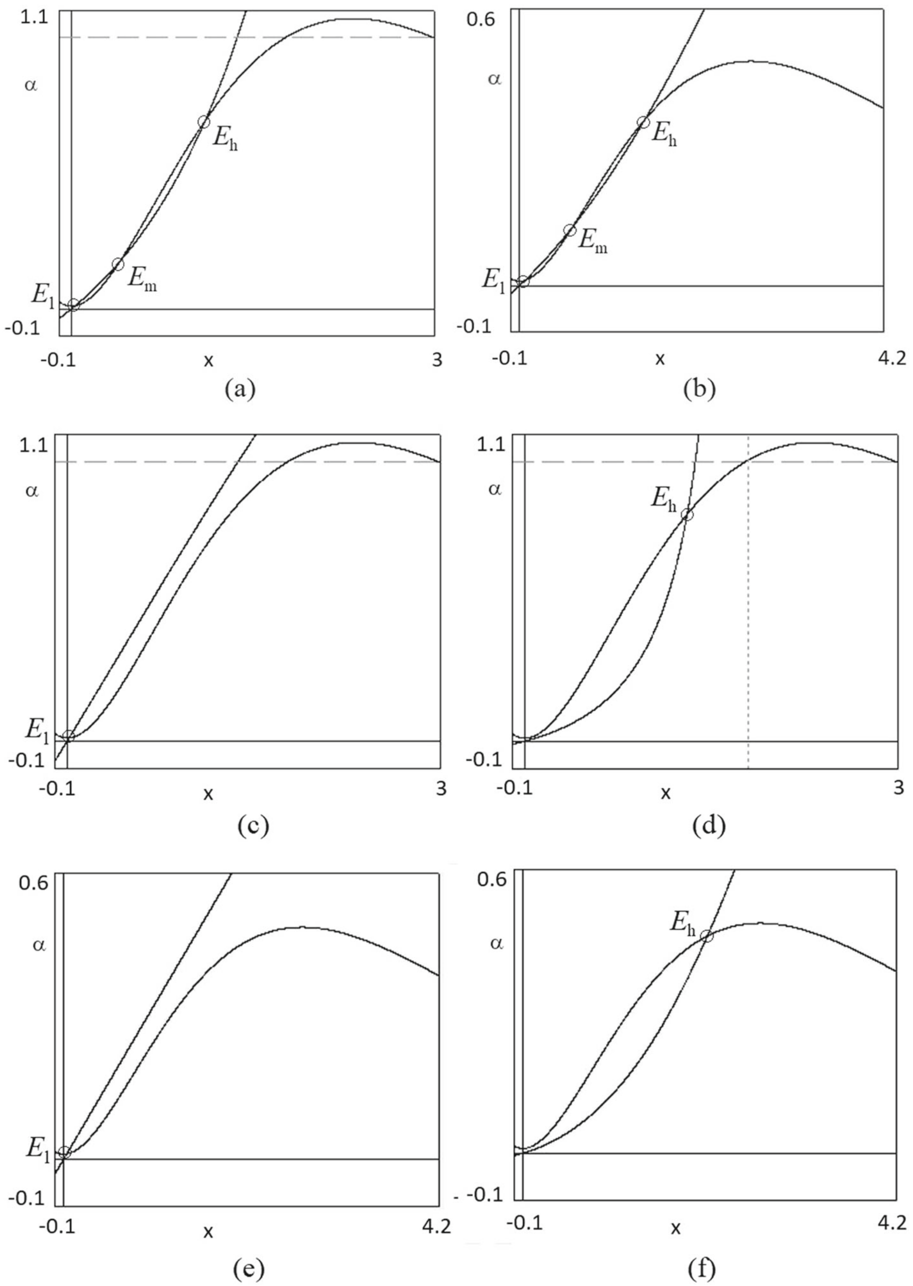


Fig. 1 The two curves (11) whose intersections define the steady states of the dynamical system. Some cases with three steady states (panels a and b), obtained for $k = 0.69$, $p = 7.3$, $m = 10$, $\mu = 0.98$, $l = 0.01$, $h = 0.86$, $\gamma = 0.3$ and $x_r = 12$ (panel a) and $k = 2.2$, $p = 3$, $m = 9.4$, $\mu = 4$, $l = 0.01$, $h = 0.75$, $x_r = 6.6$ and $\gamma = 0.8$ (panel b). Panels c-f show cases with a unique steady state. Panels c and d are obtained for the same configuration of parameters used in panel a but $x_r = 0$ (panel c) and $x_r = 20$ (panel d). Panels e and f are obtained for the same configuration of parameters used in panel b but $\gamma = 0$ (panel e) and $\gamma = 1.5$ (panel f)

The Jacobian matrix of (9) is given by

$$J(x,\alpha)=\begin{bmatrix}1-\gamma-\frac{\mu m\alpha}{x^2(1-\alpha)} & \mu\frac{m-px}{x(1-\alpha)^2}\\ \alpha'(x) & 0\end{bmatrix} \tag{12}$$

where $\alpha'(x)=\frac{1}{k}xe^{-hx}(2-hx)$. The Jacobian determinant is $\det J(x,\alpha)=-\mu\frac{m-px}{x(1-\alpha)^2}\alpha'(x)$, hence it vanishes along the lines $x=\frac{m}{p}$ and $x=\frac{2}{h}$, as $\alpha'(\frac{2}{h})=0$ The image of the line $x=\frac{2}{h}$ is a critical curve

$$LC=T\left(x=\frac{2}{h},\alpha\right)=\left(\frac{2}{h}+\gamma\left(x_r-\frac{2}{h}\right)+\frac{\mu}{2}\left(\frac{mh\alpha-2p}{1-\alpha}\right),l+\frac{4}{e^2kh^2}\right) \tag{13}$$

i.e. the line $\alpha=l+\frac{4}{e^2kh^2}$ separates the region $Z_2=\left\{(x,\alpha)\in\mathbb{R}^2|\alpha<l+\frac{4}{e^2kh^2}\right\}$, whose points have two rank-1 preimages, from the complementary region Z_0, whose points have no preimages.

The folding action of LC, which leads different points to be mapped into the same image, can be seen as an extreme form of overshooting. As argued in the previous section, the map T is non-invertible because of the presence of the backward bending portion of the preference curve $\alpha(x)$, i.e. in the model proposed in this paper, non-invertibility is due to the rejection of a previously abused good. In other words, after an excessive consumption of the good considered, a strong fall of consumption is expected in the next time period, and this can be seen as an intuitive economic explanation of the role of LC in the global dynamics of the map T.

Instead, the image of the line $x=\frac{m}{p}$ is a single point

$$T(x=\frac{m}{p},\alpha)=\left(\frac{m}{p}-\mu p+\gamma\left(x_r-\frac{m}{p}\right),\frac{m^2}{kp^2}e^{-h\frac{m}{p}}+l\right)$$

i.e. the whole line is "focalized" by T into a single point, that we shall denote as Q_{-1}. Indeed, by using the terminology introduced in (Bischi et al., 1999), we can say that the line $x=\frac{m}{p}$ is a prefocal line of T^{-1}, as explained in the next subsection.

4.2 Focal point and prefocal curves

Let us consider a two-dimensional map with at least a component not defined in the whole plane due to the presence of a denominator which can vanish. For example, the first component of the map (9) has a denominator $D(x,\alpha)=x(\alpha-1)$ vanishing along the lines $x=0$ and $\alpha=1$, denoted as the *set of nondefinition* of the map T

$$\delta_s=\{(x,\alpha)\in\mathbb{R}^2|D(x,\alpha)=0\}. \tag{14}$$

If we consider a smooth simple arc γ transverse to δ_s, its image $T(\gamma)$ is in general made up of two disjoint unbounded arcs, but a different situation may occur if the point where γ intersects δ_s is such that not only the denominator but also the numerator vanishes in it, as it occurs for the map (9) in the point $Q=(0,0)$. In this case the image curve $T(\gamma)$ may be bounded, and the following definition of *focal point* and *prefocal curve* can be given (see Bischi et al. 1999):

Definition *A point Q is a focal point for the map T if at least one component of T takes the form 0/0 in Q and there exist smooth simple arcs γ through Q such that their image $T(\gamma)$*

is finite. The set of all the finite images of Q computed along different arcs γ through Q is the prefocal set δ_Q.

Following (Bischi et al., 1999), let us consider the first component of the map in the form $\frac{N(x,\alpha)}{D(x,\alpha)}$, and let $Q = (0, 0)$ be a simple root of the algebraic system

$$N(x, y) = 0, \quad D(x, y) = 0$$

We recall that Q is simple if $\overline{N}_x \overline{D}_y - \overline{N}_y \overline{D}_x \neq 0$, where $\overline{N}_x = \frac{\partial N}{\partial x}(Q)$ and analogously for the other partial derivatives. In this case the line $\alpha = \alpha(0)$ is the corresponding prefocal curve, where $\alpha(x)$ is the preference function (10), and a one-to-one correspondence is defined between the point $(x, \alpha(0))$, in which $T(\gamma)$ crosses δ_Q, and the slope m of γ in Q, given by

$$m \to (x(m), \alpha(0)), \qquad \text{with} \qquad x(m) = (\overline{N_x} + m\overline{N_y})/(\overline{D_x} + m\overline{D_y}) \tag{15}$$

and

$$(x, \alpha(0)) \to m(x) \qquad \text{with} \qquad m(x) = (\overline{D_x}x - \overline{N_x}) \;/\; (\overline{N_y} - \overline{D_y}x). \tag{16}$$

From the definition of the prefocal curve, it follows that at least one inverse of the map T exists, say T^{-1}, such that all the points of δ_Q are mapped by T^{-1} into the focal point Q, i.e. $T^{-1}\left(\delta_Q\right) = Q$. Hence T^{-1} is not locally invertible in the points of δ_Q, being it a many-to-one map, and this implies that its Jacobian cannot be different from zero along δ_Q. Roughly speaking, a *prefocal curve* is a set of points for which at least one inverse exists which maps (or " focalizes") the whole set into a single point, called *focal point*. From the relations (15), (16) it follows that different arcs γ_j, passing through a focal point Q with different slopes m_j, are mapped by T into bounded arcs $T(\gamma_j)$ crossing δ_Q in different points $\left(x(m_j), \alpha(0)\right)$. Let δ_Q be a prefocal curve whose corresponding focal point is Q. Then each point sufficiently close to δ_Q has its rank-1 preimage in a neighborhood of the focal point Q, and if an arc ω crosses δ_Q in two distinct points, say $(x_1, \alpha(0))$ and $(x_2, \alpha(0))$ then its preimage $T^{-1}(\omega)$ must include a loop with double point in Q, as shown in the qualitative picture in Fig. 2.

In the map (9), the first component can be written as

$$x_{t+1} = \frac{N(x_t, \alpha_t)}{D(x_t, \alpha_t)} = \frac{(1-\alpha_t)x_t^2 - \mu p x_t + \mu m \alpha_t + \gamma x_t(x_r - x)(1-\alpha_t)}{x_t(1-\alpha_t)}$$

and it becomes 0/0 in $Q = (0, 0)$ and $R = \left(\frac{m}{p}, 1\right)$. These are both focal points, with corresponding prefocal curves

$$\delta_Q = \{(x, \alpha)\,|\alpha = l\}$$

and

$$\delta_R = \left\{(x, \alpha)\,|\alpha = \frac{m^2}{kp^2}e^{-\frac{hm}{p}} + l\right\}$$

respectively. The one-to-one relations (16) between slope s (through the focal point) and position $x(s)$ along the corresponding prefocal line are given by

$$x(s) = \mu m s - \mu p + \gamma x_r$$

for the focal point Q, and

$$x(s) = \frac{(1-\gamma)m - \mu p^2 - 3\gamma p x_r}{p} - \frac{\mu p^2}{m s}$$

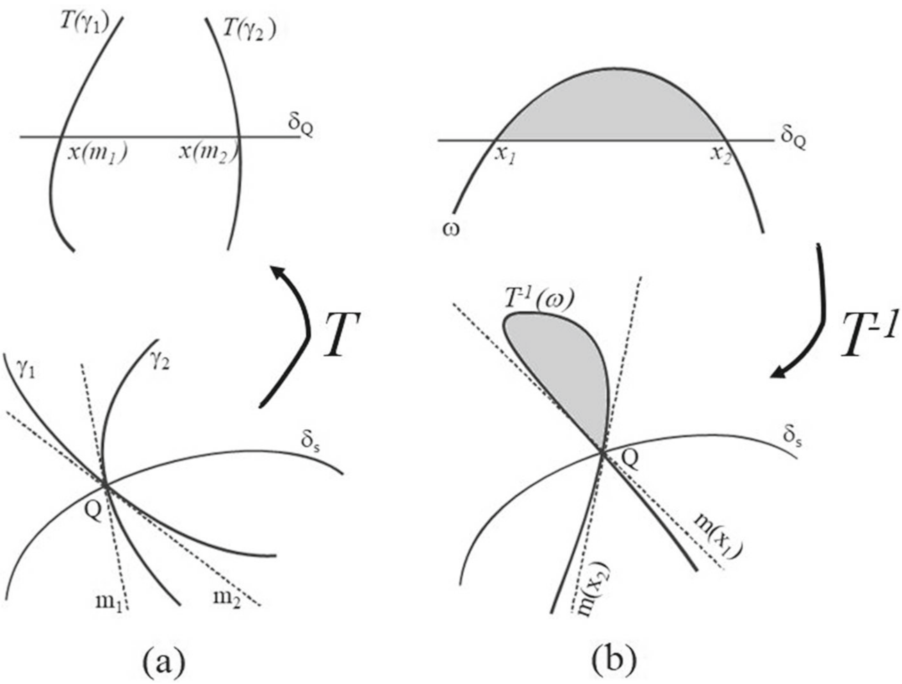


Fig. 2 Panel (a). Arcs through a focal point Q with different slopes are mapped into arcs crossing through δ_Q in different points. Panel (b). A preimage of an arc crossing through the prefocal line δ_Q into distinct points, given by a loop with double point in the focal point Q

for the focal point R. The presence of these focal points and corresponding prefocal curves has important effects on the geometrical and dynamical properties of the dynamical system considered. In fact, a contact of an arc ω with a prefocal curve gives rise to important qualitative changes in the shape of the preimages $T_j^{-1}(\omega)$, and when ω is an arc belonging to a basin boundary $\mathcal{F}$, the qualitative modifications of the preimages $T_j^{-1}(\omega)$ of ω, due to a tangential contact of ω with a prefocal curve can be particularly important for the global structure of the basin boundary. In fact, as $\mathcal{F}$ is backward invariant, i.e. $T^{-1}(\mathcal{F}) = \mathcal{F}$, if ω is an arc belonging to $\mathcal{F}$, then all its preimages of any rank must belong to $\mathcal{F}$. This implies that if a portion ω of $\mathcal{F}$ crosses a prefocal curve in two points, then the basin boundary must include loops, denoted as " lobes", somewhere along the basin boundary. As we shall see in the next section, this occurrence, together with the contacts and intersections of basin boundaries with critical curves LC, constitute the basic mechanisms leading to the involved structures of the basins of attraction.

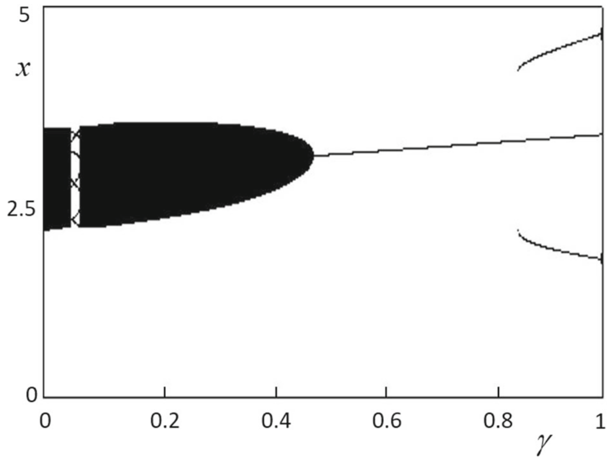
5 Global numerical explorations and contact bifurcations

In this section we use numerical methods to study the effects of the behavioral parameters γ and x_r on the long-run dynamics of consumption $x(t)$ according to the adaptive model (9) with consumers' preference adjustment function (10). We consider the fixed set of parameters $m = 10$, $p = 2.5$, $\mu = 0.98$ and $k = 1$, $h = 0.86$, $l = 0.01$ already considered in (Bischi

& Tramontana, 2023), that can be seen as a benchmark case of our model, obtained with $\gamma = 0$, i.e. without the behavioral term. In Fig. 3 a bifurcation diagram is shown, obtained with fixed reference consumption level $x_r = 5$ and bifurcation parameter γ varying in the range [0, 1]. For $\gamma = 0$ three steady states exist, given by $E_l = \left(x_l^*, \alpha_l^*\right) = (0.05, 0.01)$, saddle point, $E_m = \left(x_m^*, \alpha_m^*\right) = (0.27, 0.07)$, unstable node, $E_h = \left(x_h^*, \alpha_h^*\right) = (2.84, 0.71)$, unstable focus, and as shown in (Bischi & Tramontana, 2023) a stable closed invariant curve exist around E_h (which is a rational steady state) on which the long-run dynamics of the model are characterized by self-sustained bounded oscillations. This means that for this parameters' constellation the boundedly rational consumers described by the model do not learn to be rational, however their consumptions are confined around the rational choice even if exhibit bounded oscillations. We now assume that a reference group of consumers exists, characterized by a level of consumption $x_r = 5$ of the good considered, which is higher than the rational one (and even higher than the upper portion of the oscillations) and we numerically explore the effects of the increasing influence of the reference group consumption on the consumer described in the model, measured by increasing values of the imitation parameter γ. As can be seen in the bifurcation diagram in Fig. 3, as γ is increased beyond the value $\gamma \simeq 0.5$ the upper steady state becomes stable through a supercritical Neimark-Sacker bifurcation. We stress that for $\gamma = 0.5$ the steady state $E_h = (3.10, 0.68)$ is unique (E_l and E_m just disappeared) and the modulus of their complex conjugate eigenvalues is 0.99, i.e. it just gained local stability. It can also be noticed that, as expected, the steady state consumption is higher than the rational one, due to the tendency to imitate the consumption habits of the reference group. As explained in (Bischi & Tramontana, 2022), the steady state consumption now represents a compromise between a consumer's preferences and the higher consumption of the reference group. However, it is worth remarking that increasing γ leads to the stability of the steady state, whereas the results given in (Bischi & Tramontana, 2022) for the model with constant preferences, show that increasing values of γ generally have a destabilizing role. This is due to the fact that in the model considered in this paper the consumer's preference is adapted along the evolutive process, and the steady state is located in the decreasing portion of the preference curve, i.e. the portion characterized by the consumption saturation effect. So, on the one side the personal preferences suggest consuming less in time periods following excessive consumptions, thus leading to oscillations, whereas the tendency to imitate the reference group suggests consuming more. Hence increasing values of γ reinforce the latter effect and eliminates oscillations.

How strong is the stability of this steady state, i.e. how far from it can consumption be displaced by an exogenous shock and be sure that you will return to it? As the model considered is nonlinear, an answer to this question requires a study of the basin of attraction. This is shown in the left panel of Fig. 4, obtained for $\gamma = 0.5$, where the white region represents the set of initial conditions that generate trajectories converging to the stable steady state, whereas the points in the grey region represent initial conditions whose trajectories are unfeasible, as they involve negative values of x. We remark that feasible values of x should be limited to the range $(0, m/p) = (0, 4)$, however, as we will explain in a moment, important dynamic changes in the feasible region can only be explanted by a larger view, involving values of $x > m/p$ as well. Indeed, in the situation shown in the left panel of Fig. 4, it can be noticed that the basin of attraction of the stable steady state is a simply connected set. However, a portion of the grey basin, in the upper-right portion of the picture, is quite close the the critical line LC, represented by the horizontal line (13) of equation $\alpha = l + \frac{4}{e^2 k h^2}$, that separates the region Z_0 (formed by the points with no preimages) from Z_2 (the points with two distinct preimages). If the grey region has a contact with the line LC, after which a portion of the grey basin enters Z_2, then new preimages of the grey basin will be created

Fig. 3 Bifurcation diagram obtained with the fixed set of parameters $m = 10$, $p = 2.5$, $\mu = 0.98$, $k = 1$, $h = 0.86$, $l = 0.01$ and $x_r = 5$ with bifurcation parameter γ.



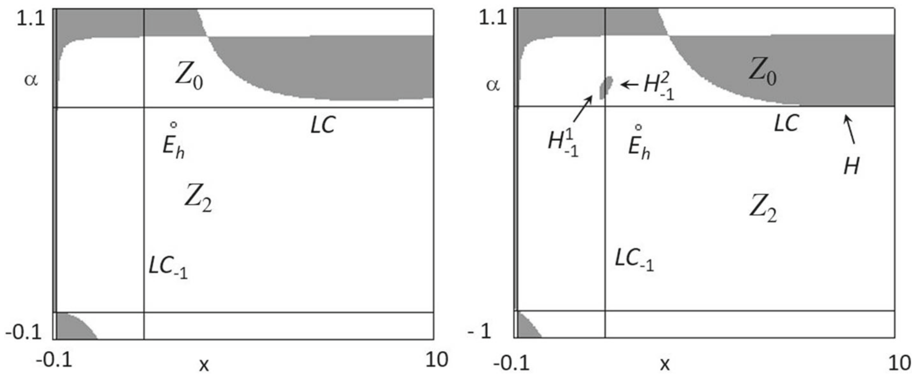


Fig. 4 Phase portraits of the model (9) with preference function (10). Panel a: the same parameters as in Fig. 3 and $\gamma = 0.5$; panel b: $\gamma = 0.68$

belonging to the gray basin as well. This is shown in the right panel of Fig. 4, obtained with a slightly larger value of the parameter γ, namely $\gamma = 0.68$. At this stage a portion of the grey basin crosses the critical line LC and its portion below LC, denoted by H in the picture, belongs to region Z_2. So two preimages of it exist, say H^1_{-1} and H^2_{-1}, located along opposite sides of the vertical line LC_{-1} of equation $x = \frac{2}{h}$, and merging along it. Following the terminology introduced by (Mira et al., 1996), this is a *hole* (or *lake*) of the basin of unfeasible trajectories nested inside the basin of the stable steady state. We can say that the grey basin of unfeasible trajectories includes a non connected component, or that the basin of the stable steady state becomes multiply connected after the contact bifurcation. As stressed above, it is important to remark that even if the hole is inside the feasible region and close to the steady state, the contact bifurcation which explains its creation occurs quite far, outside of the feasible region $x \in (0, m/p)$.

No further portions of the grey basin exist because this first hole is entirely included inside the zone Z_0 (above LC) whose points have no preimages. However, we can notice that a portion of this grey hole, whose points generate unfeasible trajectories, is quite close to the critical line LC. Indeed, if the parameter γ is further increased, so that the hole enlarges until it has a contact with LC, then a portion of the hole will enter the zone Z_2 and new preimages will be created belonging to the basin of unfeasible initial conditions. This is shown in the left panel of Fig. 5, obtained for $\gamma = 0.75$, where a portion of H_{-1} is inside Z_2. Again, this implies that it has two preimages, say H^1_{-2} and H^2_{-2} located at opposite sides with respect to the vertical line LC_{-1} and merging along it. But now, differently from the case shown in Fig. 4, the new hole H_{-2} has a portion belonging to Z_2 which in turn generates new preimages, say H^1_{-3} and H^2_{-3} etc. So more and more non connected portions of the grey basin proliferate inside the multiply connected basin of the stable steady state. Of course, this leads to more uncertainty,[3] about the fate of the trajectories following an exogenous shock in the consumption level, because some holes are close to the steady state, implying that some big shocks are recovered going back to the steady state, smaller ones may generate time evolutions of the consumption levels which never return to the steady state level.

In the meantime we can notice that some portions of the grey basin cross the prefocal lines δ_Q and δ_R thus giving rise to lobes issuing from the focal points Q and R respectively. These are clearly seen in Fig. 5.

However, the situation becomes more and more complicated if γ is further increased, as can be deduced from the bifurcation diagram shown in Fig. 3. In fact, the steady state E_h^* remains locally asymptotically stable for γ increasing up to $\gamma = 1$. On the basis of a bifurcation diagram obtained by generating, for each value of the bifurcation parameter, a trajectory starting from an initial condition taken in a neighborhood of the steady state, as usually done by commercial software programs, one may conclude that the long run evolution of the dynamical system is characterized by convergence to the unique steady state for values of the parameter γ up to $\gamma \simeq 1$. By contrast, by using different initial conditions we can see that as γ is increased above $\gamma \simeq 0.8$ a coexisting attractor exist, a stable cycle of period 2. Our numerical explorations of this dynamical system show a situation of great uncertainly about the kind of long-run evolution of the consumption levels, so the sequence of dynamic scenarios we are going to show in the following is rather emblematic. For example, the right panel of Fig. 5 shows three basins represented by three different colors: the white basin of the locally asymptotically stable steady state, the grey basin of unfeasible evolutions, the red basin of the stable cycle of period two. The basins are quite intermingled, with non connected portions created by contacts with the critical line and lobes issuing from the focal points Q and R created by contacts of basins' boundaries with the two prefocal lines.

Another dynamic scenario is shown in Fig. 6, where the role of the two focal points becomes more and more evident. Such dynamic situations reveal a very high uncertainty, even if the presence of a unique stable steady state (E_h) may suggest an apparently innocuous case according to the usual analysis of dynamical systems that represent economic systems. For instance, by further increasing the value of γ, the cycle of period two undergoes a Neimark-Sacher bifurcation at $\gamma \simeq 0.997$ originating two closed invariant curves, clearly visibile in Fig. 6, obtained with a value of γ slightly higher than one ($\gamma = 1.022$). Its basin of attraction is made up by the points in red. Moreover, a stable cycle of period four (basin in green) is also present, forming a scenario with three coexisting attractors and the global structures of the basins of attraction is even more complicated.

The dynamic scenarios shown in this section give us the opportunity to recall how a study limited to a local stability analysis might be misleading when dealing with nonlinear dynamical systems.

[3] From hereon, when we talk about *uncertainty* we refer to the long-run behavior of the system, and not to a stochastic element.

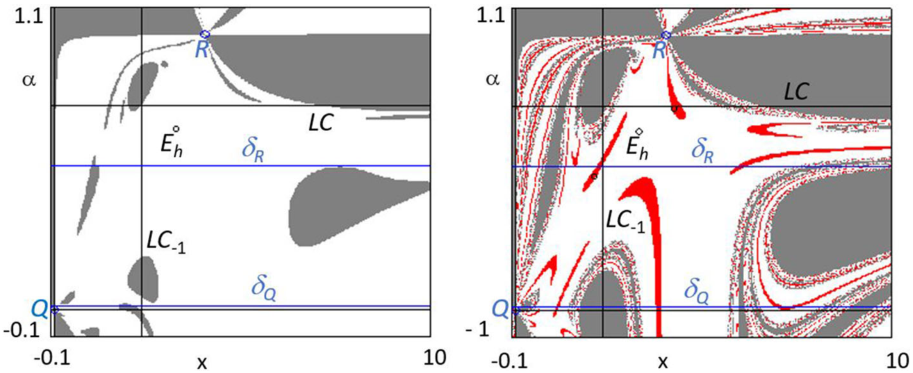
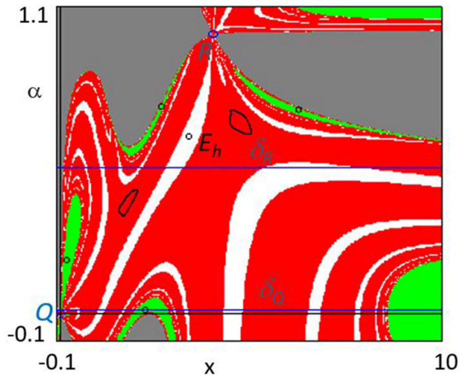


Fig. 5 Phase portraits of the model (9) with preference function (10). Panel (a): The same parameters as in Fig. 3 and $\gamma = 0.75$; Panel (b): $\gamma = 0.85$

Fig. 6 The same parameters as in Fig. 3 and $\gamma = 1.022$



6 Conclusions

In this paper a behavioral component has been introduced in a dynamic model which describes the repeated choices of a boundedly rational consumer with preferences which are adaptively updated over time. The component introduced describes the influence of the consumption of a reference group (usually of peers) that the consumer can be tempted to imitate in order to feel more integrated in the group.

So, the quantity of the good that the boundedly rational consumer decides to buy over time is conditioned by two factors that may be in contrast and hence create uncertainty about consumption decisions: the consumer's own preferences (or tastes) and the consumption levels of the reference group.

Indeed, the dynamic scenarios analyzed in this paper, mainly numerically due to the analytical difficulties related to high nonlinearity of the model, show that the presence of these two contrasting effects create several situations of coexistence of different attractors, each with its own basin of attraction, and the basins are quite intermingled, this giving strong path dependence with great uncertainty about the final outcome of the time evolution generated by different initial conditions (e.g. due to the effect of even small exogenous shocks). We have also showed another, at first sight counterintuitive, result, proving exactly the opposite of what we have just said: that is, when the consumer displays more than one rational steady state, the presence of the behavioral component may break the coexistence and leave only the steady state which is closest to the reference consumption.

The global numerical explorations of the basins and their global (or contact) bifurcations leading to complicated topological structures of the basins' boundaries, has been guided by a theoretical (and analytical) study of some singularities that characterize the global properties of the dynamical system considered, such as critical curves and prefocal lines.

Generally, the analytical expressions of these curves are not known, whereas in this case it has been possible to analytically describe them. This has allowed us to characterize some global bifurcations that may lead to disconnected basins of attractions and finger-shaped lobes issuing from single points of the phase plane. These dynamic scenarios, together with their economic consequences, clearly show the importance of a global analysis of nonlinear dynamical systems, which can often be performed only through an heuristic methods obtained by a combination of analytical, geometrical and numerical methods. In fact, a study limited to an analytical study of the local stability and bifurcations, based on the linear approximation of the model around the steady states, may sometimes be quite incomplete and even misleading, as the example considered in this paper clearly shows. Indeed, this stream of literature, see e.g. (Leijonhufvud, 1973) or (Dohtani et al., 2007), stresses the fact that nonlinear dynamic models may have the property that small perturbations are recovered in as far as they are confined inside the basin of attraction of a locally stable steady state, whereas larger perturbations lead to time evolutions that further depart from the steady state and go to the coexisting attractor in the long run, a situation that has been called "corridor stability". Instead, the results given in this paper show a quite different situation when non-connected basins of attraction, formed by disjoint (and sometimes far) portions of the basin, exist. In fact, the presence of non-connected basins can be described by saying that a small perturbation can be recovered by the endogenous dynamics of the evolutive model, a medium-size perturbation may lead to a different attractor, i.e., a different long run evolution, whereas a larger perturbation may be recovered leading back the system to the original attractor.

This work can be extended in several directions. We can test how general our result can be by using different functions explaining how past consumption may influence the preference for the good itself. We can also introduce more goods and consider how the consumption of each of them may also influence the preferences for the others. Finally, an interesting extension is the one where the reference quantity becomes endogenous and changes over time. We aim at exploring these scenarios in our future work.

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Declarations

Conflict of interest: All authors declare that they have no conflict of interest.

Appendix: Noninvertible maps and critical sets

In this appendix we introduce some general concepts, notations and definitions about the mathematical theory of discrete dynamical systems, in particular those which are represented by the iteration of noninvertible maps.

A map $T : S \to S$, $S \subseteq \mathbb{R}^n$, defined by $\mathbf{x}' = T(\mathbf{x})$, transforms a point $\mathbf{x} \in S$ into a unique point $\mathbf{x}' \in S$. The point $\mathbf{x}'$ is called the *rank-1 image* of $\mathbf{x}$, and a point $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{x}'$ is a *rank-1 preimage* of $\mathbf{x}'$. If $\mathbf{x} \neq \mathbf{y}$ implies $T(\mathbf{x}) \neq T(\mathbf{y})$ for each $\mathbf{x}, \mathbf{y}$ in S, then T is an *invertible map* in S, because the inverse mapping $\mathbf{x} = T^{-1}(\mathbf{x}')$ is uniquely defined; otherwise T is a said to be a *noninvertible map*, because points $\mathbf{x}$ exist that have several rank-1 preimages, i.e. the inverse relation $\mathbf{x} = T^{-1}(\mathbf{x}')$ is multivalued. So, noninvertible means " many-to-one", that is, distinct points $\mathbf{x} \neq \mathbf{y}$ may have the same image, $T(\mathbf{x}) = T(\mathbf{y}) = \mathbf{x}'$. Geometrically, the action of a noninvertible map can be expressed by saying that it " folds and pleats" the space S, so that distinct points are mapped into the same point. This is equivalently stated by saying that several inverses are defined in some points of S, and these inverses " unfold" S. For a noninvertible map, S can be subdivided into regions Z_k, $k \geq 0$, whose points have k distinct rank-1 preimages. Generally, for a continuous map, as the point $\mathbf{x}'$ varies in $\mathbb{R}^n$, pairs of preimages appear or disappear as it crosses the boundaries separating different regions. Hence, such boundaries are characterized by the presence of at least two coincident (merging) preimages. This leads us to the definition of the *critical sets*, one of the distinguishing features of noninvertible maps see 1980; 1996:

Definition The *critical set* CS of a continuous map T is defined as the locus of points having at least two coincident $rank - 1$ preimages, located on a set CS_{-1}, called *set of merging preimages*.

The critical set CS is generally formed by $(n-1)$-dimensional hypersurfaces of $\mathbb{R}^n$, and portions of CS separate regions Z_k and Z_{k+2} (this is the standard occurrence for continuous maps). It is the n-dimensional generalization of the notion of local minimum or local maximum value of a one-dimensional map, and of the notion of *critical curve* LC of a noninvertible two-dimensional map. The set CS_{-1} is the generalization of local extremum point of a one-dimensional map, and of the *fold curve* LC_{-1} of a two-dimensional noninvertible map. For example, if a one-dimensional differentiable map $f(x)$ has a local maximum point c_{-1}, then the first derivative vanishes, $f'(c_{-1}) = 0$ and the corresponding maximum value $c = f(c_{-1})$ (called critical point) is a "turning point" that separates the real line into the two subsets: Z_k (above c) and Z_{k+2} (below c), such that two distinct points, say x_1 and x_2, located symmetrically with respect to its preimage c_{-1}, are mapped into the same point $x' < c$. We can consider the range of the map f formed by the superposition of two half-lines, joined at the critical point c, and on each of these half-lines a different inverse is defined. This point of view gives a geometric visualization of the critical point c as the point in which two distinct inverses merge. The action of the inverses, say $f^{-1} = f_1^{-1} \cup f_2^{-1}$, causes an unfolding of the range by mapping c into c_{-1} and by opening the two half-lines one on the right and one on the left of c_{-1}. Another interpretation of the folding action of a critical point is the following. Since $f(x)$ is increasing on the left of c_{-1} and decreasing on the right (being it a maximum), the application of the map $f(x)$ is orientation preserving if $x < c_{-1}$, orientation reversing for $x > c_{-1}$. This suggests that an application of f to a segment including the point c_{-1} folds it, the folding point being the critical point c.

Let us now consider the case of a continuous two-dimensional map $T : S \to S$, $S \subseteq \mathbb{R}^2$, defined by

$$T : \begin{cases} x_1' = T_1(x_1, x_2) \\ x_2' = T_2(x_1, x_2) \end{cases},$$

If we solve the system of these two equations with respect to the unknowns x_1 and x_2, then, for a given $\left(x_1', x_2'\right)$, we may have several solutions, representing rank-1 preimages (or backward iterates) of $\left(x_1', x_2'\right)$, say $(x_1, x_2) = T^{-1}\left(x_1', x_2'\right)$, where T^{-1} may be a multivalued relation. In this case we say that T is noninvertible, and the critical set (formed by critical curves, denoted by LC from the French " Ligne Critique") constitutes the set of boundaries that separate regions of the plane characterized by a different number of rank-1 preimages. According to the definition, along LC at least two inverses give merging preimages, located on LC_{-1} see (Gumowski & Mira, 1980; Mira et al., 1996). For a continuous and (at least piecewise) differentiable noninvertible map of the plane, with jacobian matrix

$$\mathbf{DT} = \begin{bmatrix} \partial T_1/\partial x_1 & \partial T_1/\partial x_2 \\ \partial T_2/\partial x_1 & \partial T_2/\partial x_2 \end{bmatrix}$$

LC_{-1} is included in the set where the jacobian determinant $\det \mathbf{DT}(x_1, x_2)$ changes sign, since T is locally an orientation preserving map near points (x_1, x_2) such that $\det \mathbf{DT}(x_1, x_2) > 0$ and orientation reversing if $\det \mathbf{DT}(x_1, x_2) < 0$. In order to explain this point, let us recall that when an affine transformation $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where $A = \left\{a_{ij}\right\}$ is a 2×2 matrix and $\mathbf{b} \in \mathbb{R}^2$, is applied to a plane figure, then the area of the transformed figure grows, or shrinks, by a factor $\rho = |\det A|$, and if $\det A > 0$ then the orientation of the figure is preserved, whereas if $\det A < 0$ then the orientation is reversed.

If the map is continuously differentiable then the change of the sign of DT occurs along points where DT vanishes, thus giving the characterization of the fold line LC_{-1} as the locus where the jacobian vanishes. In fact, in any neighborhood of a point of LC_{-1} there are at least two distinct points which are mapped by T in the same point. Accordingly, the map is not locally invertible in points of LC_{-1}, and the jacobian determinant vanishes as a consequence of the implicit function theorem. Many of the considerations made above, for 1-dimensional and 2-dimensional noninvertible maps, can be generalized to n-dimensional ones, even if their visualization becomes more difficult. This provides an easy method to compute the critical set for continuously differentiable maps: from the expression of the jacobian determinant one computes the locus of points at which it vanishes, then the set obtained after an application of the map to these points is the critical set CS.

A *discrete-time dynamical system*, defined by the difference equation

$$\mathbf{x}_{t+1} = T(\mathbf{x}_t)$$

can be seen as the result of the repeated application (or *iteration*) of a map T. Indeed, the point $\mathbf{x}$ represents the state of a system, and T represents the " unit time advancement operator" $T : \mathbf{x}_t \to \mathbf{x}_{t+1}$. Starting from an *initial condition* $\mathbf{x}_0 \in S$, the iteration of T inductively defines a unique *trajectory*

$$\tau(\mathbf{x}_0) = \left\{\mathbf{x}_t = T^t(\mathbf{x}_0), t = 0, 1, 2, ...\right\},$$

where T^0 is the identity map and $T^t = T(T^{t-1})$. As $t \to +\infty$, a trajectory may diverge, or it may converge to a fixed point of the map T, i.e. a point $\overline{\mathbf{x}}$ such that $T(\overline{\mathbf{x}}) = \overline{\mathbf{x}}$, or it may asymptotically approach another kind of invariant set, such as a periodic cycle, or a closed invariant curve or a more complex attractor, for example a so called chaotic attractor (see e.g. (Devaney, 1987),[20] and (Medio & Lines, 2001)). We recall that a set $A \subset \mathbb{R}^n$ is *invariant* for the map T if it is mapped onto itself, $T(A) = A$, i.e. if $x \in A$ then $T(x) \in A$. A closed invariant set A is an *attractor* if the following two conditions hold: (i) for every neighborhood W of A there exists a neighborhood V of A such that $T^t(V) \subset W\ \forall t \geq 0$; (ii) a neighborhood U of A exists such that $T^t(\mathbf{x}) \to A$ as $t \to +\infty$ for each $\mathbf{x} \in U$.

The *basin* of an attractor A is the set of all points that generate trajectories converging to A

$$\mathcal{B}(A) = \left\{\mathbf{x} | T^t(\mathbf{x}) \to A \text{ as } t \to +\infty\right\} \tag{17}$$

Let $U(A)$ be a neighborhood of an attractor A whose points converge to A. Of course $U(A) \subseteq \mathcal{B}(A)$, and also the points that are mapped into U after a finite number of iterations belong to $\mathcal{B}(A)$. Hence, the basin of A is given by

$$\mathcal{B}(A) = \bigcup_{n=0}^{\infty} T^{-n}(U(A)) \tag{18}$$

where $T^{-1}(\mathbf{x})$ represents the set of the rank-1 preimages of $\mathbf{x}$ (i.e. the points mapped into x by T), and $T^{-n}(x)$ represents the set of the rank-n preimages of x (i.e. the points mapped into x after n applications of T).

Let $\mathcal{B}$ be a basin of attraction and $\partial\mathcal{B}$ its boundary. From the definition it follows that $\mathcal{B}$ is trapping with respect to the forward iteration of the map T and invariant with respect to the backward iteration of all the inverses T^{-1}. In other words, points belonging to $\partial\mathcal{B}$ are mapped into $\partial\mathcal{B}$ both under forward and backward iteration of T. This implies that if an unstable fixed point or cycle belongs to $\partial\mathcal{B}$ then $\partial\mathcal{B}$ must also contain all of its preimages of any rank. Moreover, if a saddle-point, or a saddle-cycle, belongs to $\partial\mathcal{B}$, then $\partial\mathcal{B}$ must also contain the whole *stable set*, i.e. all the points that generate trajectories that converge to the saddle fixed point as $t \to +\infty$ (see (Gumowski & Mira, 1980; Mira et al., 1996) and (Robinson, 2012)).

A problem that often arises in the study of nonlinear dynamical systems concerns the existence of several attracting sets, each with its own basin of attraction. In this case the dynamic process becomes path-dependent, i.e. which kind of long run dynamics characterizes the system depends on the starting condition. This question requires an analysis of the global dynamical properties of the dynamical system, that is, an analysis which is not based on the linear approximation of the map. When the map T is noninvertible, its global dynamical properties can be usefully characterized by using the formalism of critical sets, by which the folding action associated with the application of the map, as well as the " unfolding" associated with the action of the inverses, can be described. Loosely speaking, the repeated application of a noninvertible map repeatedly folds the state space along the critical sets and their images, and often this allows one to define a bounded region where asymptotic dynamics are trapped. As some parameter is varied, global bifurcations that cause sudden qualitative changes in the properties of the attracting sets can be detected by observing contacts of critical curves with invariant sets. On the other hand, the repeated application of the inverses " repeatedly unfold" the state space, so that a neighborhood of an attractor may have preimages far from it, thus giving rise to complicated topological structures of the basins, that may be formed by the union of several (even infinitely many) non connected portions. In fact, in order to study the extension of a basin and the structure of its boundaries one has to consider the properties of the inverse relation T^{-1}.The route to more and more complex basin boundaries, as some parameter is varied, is characterized by global bifurcations, also called contact bifurcations, due to contacts between the critical set and the invariant sets that form the basins' boundaries. So, it is clear that the properties of the inverses are important in order to understand the structure of the basins and the main bifurcations which change their qualitative properties. In the case of noninvertible maps, the multiplicity of preimages may lead to basins with complex structures, such as multiply connected or non connected sets, sometimes formed by (even infinitely) many non connected portions (see (Mira et al., 1996), ch.5, (Abraham et al., 1997),ch.5).

Several examples of two-dimensional noninvertible maps that have non connected basins can be found in the recent literature, see e.g. (Puu, 2000; Bischi & Kopel, 2001; Bischi et al., 2003; Agliari et al., 2002) and (Agliari et al., 2004). Examples in three dimensions are given in (Bischi et al., 2001) and (Agliari et al., 2002).

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