ORIGINAL RESEARCH

Dynamical inspection game with continuous strategies

Ugo Merlone[1] · Giovanni Orlando[2] · Arianna Dal Forno[3]

Abstract

Most contributions on the inspection game concern arm control and disarmament; recently some contributions have considered organizational settings. We consider an inspection game where a principal chooses to inspect or not inspect and an agent simultaneously can either work or shirk. Combined payoffs are maximized when the principal does not inspect and the agent works while the unique Nash equilibrium of the stage game is in mixed strategies with positive probabilities of inspecting/shirking. To overcome this difficulty we introduce a continuous action version of the inspection game which extends the original formulation and discuss the existence of the Nash equilibria in pure strategies depending on the convexity of the cost functions we consider. Then, as most of the interactions in organizations develop over time, we propose a dynamic model with adaptive adjustment. We address some characteristics of the dynamic behavior of the game and the bifurcations observed, through both analytical and numerical methods. For the dynamical game we determine the fixed points, and study their stability. Fixed points are related to the Nash equilibria of the continuous inspection game and the collectively optimal outcome is obtained as a fixed point that is just *virtual*. Our findings are interpreted in terms of stakeholders theory, relational contracts and negotiation.

Keywords Inspection game · Continuous actions · Discrete time dynamics · Equilibrium stability · Negotiation

1 Introduction

Inspection games are one of the many applications of Game Theory (Avenhaus et al. 2002; Norozpour and Safaei 2020). An inspection game is a mathematical model in which a player checks whether another player adheres to certain legal rules. For example, this legal behavior can be defined by an arms control treaty, and the inspectable subject has potential interest in violating it. Since the inspector's resources are usually limited, the verification can be only

✉ Ugo Merlone
  ugo.merlone@unito.it

1   Department of Psychology, Center for Logic, Language and Cognition, University of Turin, Turin,
    Italy

2   Department of Management and Organization, Vrije Universiteit Amsterdam, Amsterdam,
    The Netherlands

3   Department of Economics, University of Molise, Campobasso, Italy

partial. If an illegal action is supposed to be carried out strategically, a mathematical analysis could help design an optimal inspection scheme (Avenhaus et al. 2002).

In the systematic review on the inspection game conducted in Orlando (2022) 54 core articles been selected and divided in three main fields: criminology, mathematics and management/organizational economics.[1] Although most contributions about this interaction concern arm control and disarmament, recently some authors have considered interactions between an employer (principal) who can either inspect or not an employee (agent) who, simultaneously, chooses if either working or shirking. In this paper we consider the standard inspection game in which both players have two strategies, respectively *Inspect* or *Not inspect* for the principal, and *Work* or *Shirk* for the agent. However, since in most formalizations (Fudenberg and Tirole 1991; Nosenzo et al. 2016) the unique Nash equilibrium of the inspection game is in mixed-strategies, and mixed-strategy equilibria may be described "as steady states in a large population in which players use pure strategies but the population as a whole mimics a mixed-strategy" (Oechssler 1997), this equilibrium is not easy to interpret when considering a dyadic interaction. Furthermore, it is natural to assume that cooperative behaviors can vary continuously within a certain range (Doebeli et al. 2004). For this reason we introduce a continuous generalization of the inspection game in order to determine only pure strategy equilibria and we postpone the case with mixed-strategy equilibria to further research.

Besides the monograph (Avenhaus and Krieger 2020) which uses mostly an optimal timing under imperfect information approach, to the best of our knowledge two contributions extend the inspection game as presented in Fudenberg and Tirole (1991) in a repeated setting: Fandel and Trockel (2013) and Nosenzo et al. (2016). The first one considers an infinite repeated game in which an intervention by a third party may occur with some probability. The second contribution considers the introduction of a punishment in an experimental setting with finite repetitions. In both cases the punishment is considered as a third move by one of the parties and the strategies remain discrete. However, considering repeated interactions rather than one shot games remains important, as interactions in teams and work groups are intrinsically dynamical (Gorman et al. 2017). More generally, Thiétart and Forgues (1995) argued that, under some conditions, organizations are likely to exhibit the qualitative properties of chaotic systems. This framework has been really fruitful: for instance, according to Vibert (2004), principles of chaos theory have helped organizational theorists to analyze several aspects of organizations; a list of "organizational propositions based on chaos and complexity theory" is provided in Houry (2012); small groups have been analyzed in terms of complex systems (Arrow et al., 2000). For a recent review on teams considered as complex systems the reader may refer to Ramos-Villagrasa et al. (2018). This approach has provided insights in understanding dynamic complexity (Chia 1998; Griffin et al. 1998) and exploring non-linear causality (Flatau 1995). In order to study how the repeated interaction in discrete time evolves, we introduce an adaptive process in which the principal partially adjusts the level of inspection accuracy and the agent the effort level to be exerted, towards the best reply with naive expectations; that is, the principal assumes the current level of effort by the agent and the latter assumes the current level of inspection accuracy by the principal in order to compute their best reply in the next period.

By considering such an adaptive process, we aim to contribute to the literature with the introduction of decisions that are adjusted over time. This will help to represent the

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[1] As this field is intrinsecally interdisciplinary, Ackoff's words: "Disciplines are categories that facilitate filing the content of science. They are nothing more than filing categories. Nature is not organized the way our knowledge of it is. Furthermore, the body of scientific knowledge can, and has been, organized in different ways. No one way has ontological priority." Ackoff (1973) are particularly poignant.

Table 1 The payoff matrix of the standard inspection game

$\begin{array}{c} A \\ P \end{array}$	Shirk	Work
Not Inspect	$\begin{array}{cc} & w \\ -w & \end{array}$	$\begin{array}{cc} & w-g \\ v-w & \end{array}$
Inspect	$\begin{array}{cc} & 0 \\ -h & \end{array}$	$\begin{array}{cc} & w-g \\ v-w-h & \end{array}$

organizational environment through the lens of the inspection game with consideration of the logic that lead to the change of behavior from a self-focused point to a new perspective of joint value creation (Bridoux and Stoelhorst 2016). With such a perspective we also aim to fill a gap in the literature that sees the inspection game as a static model where the two parties interact but never adjust their responses to avoid undesirable consequences.

The structure of the paper is the following. Section 2 recalls the one-shot inspection game and its mixed-strategy equilibrium. Section 3 introduces continuous strategies in the model and includes the one-shot game as a special case. In general, unlike the original game, the existence of a Nash equilibrium is not guaranteed in the continuous game. Therefore, a thorough analysis about the existence and uniqueness is conducted in Sect. 4. Section 5 is devoted to the particular case of the continuous one-shot inspection game with quadratic costs, and to a characterization of both its interior and border Nash equilibrium. With the assumption of quadratic costs, in Sect. 6 we first propose an inspection game dynamic model with adaptive adjustment and discuss how the equilibria of the dynamical system relates to the Nash equilibria of both the original and the continuous game. Then we address some characteristics of the dynamic behavior of the model and the particular bifurcations observed, through both analytical and numerical methods. In Sect. 7 we discuss how our findings relate to the organizational literature, provide some proposals for further research, and conclude.

2 The inspection game Γ^0

According to the version of the inspection game as presented in Fudenberg and Tirole (1991), an agent A works for a principal P, hence the set of players is $I = \{P, A\}$. The agent can either *Shirk* or *Work*, while the principal can either *Not Inspect* or *Inspect*; that is, we have the sets of finite strategies $S_P^0 = \{\textit{Not Inspect, Inspect}\}$ and $S_A^0 = \{\textit{Shirk, Work}\}$. When working, the agent bears a cost $g > 0$ of fatigue, and produces output $v > 0$ for the principal; by inspecting at a cost $h > 0$, the principal can gather evidence about the agent's action. The principal pays the agent a wage $w > 0$ if there is no evidence of shirking, otherwise $w = 0$. The two players choose their strategies simultaneously. Both the inspection and the fatigue costs do not exceed the wage, that is, $w > h, g$. The payoff matrix of the standard inspection game is presented in Table 1 and defines the payoff functions $p_P : S_P^0 \times S_A^0 \to \mathbb{R}$ and $p_A : S_P^0 \times S_A^0 \to \mathbb{R}$, respectively for the principal and the agent.

In the one-shot game

$$\Gamma^0 = \left(I, \left(S_P^0, S_A^0\right), (p_P, p_A)\right)$$

there is not a pure strategy equilibrium. We can see the lack of pure strategies equilibrium through this "cycle of retaliations":

$$(Inspect, Work) \rightarrow (Not\ Inspect, Work) \rightarrow (Not\ Inspect, Shirk) \rightarrow (Inspect, Shirk)$$

and so on. Rather, there exists a unique mixed-strategy Nash equilibrium in which shirking occurs with positive probability (Nosenzo et al. 2016). The mixed-strategy Nash equilibrium is

$$\left(\left(1 - \frac{g}{w}, \frac{g}{w}\right), \left(\frac{h}{w}, 1 - \frac{h}{w}\right)\right) \tag{1}$$

where the probability of inspecting is g/w and the probability of working is $1 - h/w$. Note that the game has two strategy profiles, that is, (*Not Inspect*, *Work*) and (*Not Inspect*, *Shirk*), that are Pareto efficient. Only the former maximizes the joint payoffs although, as "optimal behavior of both players cannot be guaranteed" (Fandel and Trockel 2013, p. 496), players will not stick to it given the game theory assumptions.

Beside the model formalization, the value v of the output does not play an explicit role neither in the equilibrium nor in the analysis which follows; therefore, to keep the model meaningful, we will generally assume $v > w$ without specifying its value.

3 The continuous inspection game Γ

In order to obtain a continuous action spaces generalization of the game Γ^0 presented in Sect. 2, we assume that the principal may decide her inspection accuracy a, with $0 \leq a \leq 1$, and that inspecting costs her $c_P(a) = h \cdot a^\alpha$, where α is a positive constant and h is defined as in the previous section. Similarly, we assume that the agent decides his effort e, with $0 \leq e \leq 1$, bears a cost $c_A(e) = g \cdot e^\beta$, where β is a positive constant and g is defined above, and that the value of the output becomes contingent on the agent's effort. Finally, we introduce a side payment depending on the principal's inspection accuracy a and agent's exerted effort e. Side payment is in the form of a salary cut; with some caveats it may be interpreted as punishment, and is determined as follows

$$s := wa(1 - e). \tag{2}$$

The salary cut is increasing w.r.t. the principal's accuracy a and decreasing w.r.t. the agent's effort e. This way we obtain a game with continuous strategies $a, e \in [0, 1]$ and payoff functions

$$\pi_P(a, e) = ve - w - ha^\alpha + wa(1 - e) \tag{3a}$$
$$\pi_A(a, e) = w - ge^\beta - wa(1 - e) \tag{3b}$$

where ve is the value of the output depending on agent's effort, w is the wage, ha^α is the cost of inspection, $wa(1 - e)$ is the side payment, ge^β is the cost of effort. The continuous game can be formalized as

$$\Gamma = (I, (S_P = [0, 1], S_A = [0, 1]), (\pi_P, \pi_A)).$$

Obviously, the side payment s vanishes when the aggregate profit is considered. This way we could assume that the principal and the agent will play in a way that will first maximize the take for all and then use s as an implicit mechanism for bargaining concerning shares as suggested in Shubik (1972). The collectively optimal solution consists of the agent exerting

full effort ($e = 1$) and the principal not inspecting ($a = 0$); this solution is Pareto efficient yet still not sustainable as in game Γ^0.

Proposition 1 *The inspection game Γ^0 formalized in* Fudenberg and Tirole (1991) *is a special case of game Γ.*

Proof

- For the strategy profile (*Not Inspect*, *Shirk*) we have $a = 0$ and $e = 0$ and from (3) we obtain

$$
\begin{aligned}
\pi_P (0, 0) &= -w \\
\pi_A (0, 0) &= w
\end{aligned}
\tag{4}
$$

- For the strategy profile (*Not Inspect*, *Work*) we have $a = 0$ and $e = 1$ and from (3) we obtain

$$
\begin{aligned}
\pi_P (0, 1) &= v - w \\
\pi_A (0, 1) &= w - g
\end{aligned}
\tag{5}
$$

- For the strategy profile (*Inspect*, *Shirk*) we have $a = 1$ and $e = 0$ and from (3) we obtain

$$
\begin{aligned}
\pi_P (1, 0) &= -h \\
\pi_A (1, 0) &= 0
\end{aligned}
\tag{6}
$$

- For the strategy profile (*Inspect*, *Work*) we have $a = 1$ and $e = 1$ and from (3) we obtain

$$
\begin{aligned}
\pi_P (1, 1) &= v - w - h \\
\pi_A (1, 1) &= w - g
\end{aligned}
\tag{7}
$$

$\square$

4 Nash equilibria of the continuous inspection game

Both the principal and the agent respectively choose their accuracy and effort levels in order to maximize their profit. The existence of the Nash equilibrium in pure strategies is not guaranteed in general for all $\alpha, \beta \in \mathbb{R}_+$. In order to prove it, we will decompose the problem case by case, and summarize the results at the end of this section.[2]

4.1 Case: $(0, 0) < (\alpha, \beta) \leq (1, 1)$

Proposition 2 *In the inspection game Γ with continuous strategies and $(0, 0) < (\alpha, \beta) \leq (1, 1)$, the Nash equilibrium in pure strategies does not exist.*

Proof When $(\alpha, \beta) \leq (1, 1)$ both profit functions (3) are convex in the compact set $[0, 1]$ when considering each player's decision variable; the FOCs identify a minimum point and the maximum points might respectively be at either $a = 0$ or $a = 1$ and either at $e = 0$ or $e = 1$. However, none of the four corners of set U can be an equilibrium and we can follow the same reasoning used to illustrate the lack of pure strategy equilibria of game Γ^0. For instance, the payoff of the agent is equivalent by playing either $e = 0$ or $e = 1$ only when the principal inspects at level $a = g/w$. If the principal inspects at any level $a < g/w$, the

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agent strictly prefers shirking, and therefore the principal is better off inspecting at any level $a > g/w$. On the other hand, if the principal inspects at any level $a > g/w$, the agent prefers working, and the principal is better off inspecting less than g/w. If this holds for any level of inspection a lesser or greater than g/w then it holds also for $a = 0$ or $a = 1$ respectively. A similar reasoning applies for the other player.   □

4.2 Case: $(\alpha, \beta) \geqq (1, 1)$

Proposition 3 *In the inspection game Γ with continuous strategies and $(\alpha, \beta) \geqq (1, 1)$, the Nash equilibrium in pure strategies always exists and is unique.*

Proof In order to prove it we need to make some distinctions.

- When $(\alpha, \beta) > (1, 1)$, this is a concave 2-person game since every joint strategy lies in $[0, 1]^2$ that is convex, closed, and bounded and player's payoff functions are concave in their own strategy. Furthermore, following Rosen (1965), it is easy to see that the sufficient condition for the weighted nonnegative sum of payoffs

$$\sigma(a, e, r_1, r_2) = r_1\pi_P + r_2\pi_A, \quad r_1, r_2 \geq 0$$

to be diagonally strictly concave holds at least for $r_1 = r_2 = \bar{r} > 0$; therefore, the equilibrium point exists and is unique. The FOCs on the objective functions (3) yield respectively:

$$\frac{\partial \pi_P}{\partial a} = -\alpha h a^{\alpha-1} + w(1-e) = 0 \tag{8a}$$

$$\frac{\partial \pi_A}{\partial e} = -\beta g e^{\beta-1} + wa = 0 \tag{8b}$$

from which, as profits are concave, we derive the best reply functions

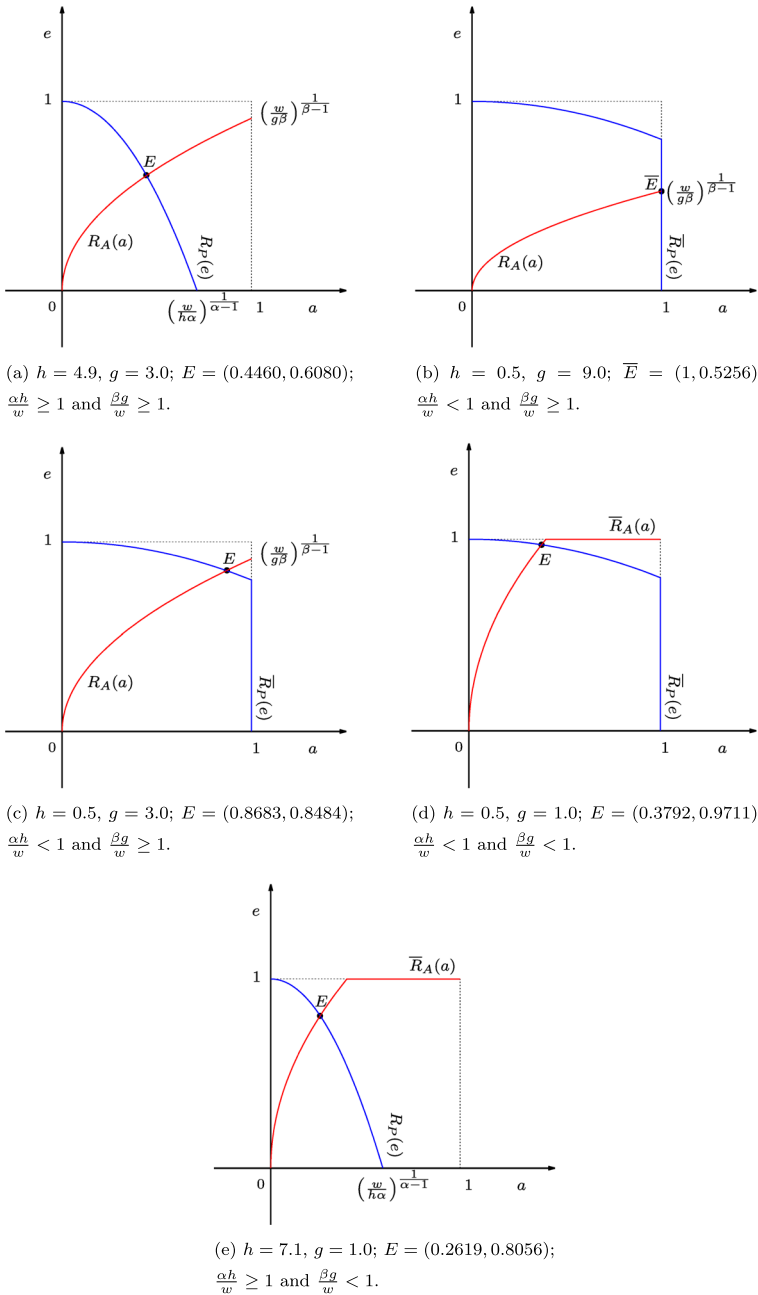$$R_P(e) = \left(\frac{w(1-e)}{\alpha h}\right)^{\frac{1}{\alpha-1}} \tag{9}$$

and

$$R_A(a) = \left(\frac{wa}{\beta g}\right)^{\frac{1}{\beta-1}} \tag{10}$$

defining the Nash equilibrium

$$\begin{cases} R_P(e^*) = a^* \\ R_A(a^*) = e^* \end{cases} \tag{11}$$

The two functions are always well defined (and solution (a^*, e^*) is feasible) when $\frac{\alpha h}{w} \geq 1$ and $\frac{\beta g}{w} \geq 1$. Otherwise, that is, when $\frac{\alpha h}{w} < 1$, depending on the other player's strategy, equilibrium a^* might not be interior to the interval $[0, 1]$ and, in order to be feasible, the constraint is binding. In this case, the principal's best reply function is:

$$\overline{R}_P(e) = \begin{cases} 1 & \text{if } 0 \leq e \leq 1 - \frac{\alpha h}{w} \\ \left(\frac{w(1-e)}{h\alpha}\right)^{\frac{1}{\alpha-1}} & \text{if } 1 - \frac{\alpha h}{w} < e \leq 1. \end{cases} \tag{12}$$

(a) $h = 4.9$, $g = 3.0$; $E = (0.4460, 0.6080)$; $\frac{\alpha h}{w} \geq 1$ and $\frac{\beta g}{w} \geq 1$.

(b) $h = 0.5$, $g = 9.0$; $\overline{E} = (1, 0.5256)$; $\frac{\alpha h}{w} < 1$ and $\frac{\beta g}{w} \geq 1$.

(c) $h = 0.5$, $g = 3.0$; $E = (0.8683, 0.8484)$; $\frac{\alpha h}{w} < 1$ and $\frac{\beta g}{w} \geq 1$.

(d) $h = 0.5$, $g = 1.0$; $E = (0.3792, 0.9711)$; $\frac{\alpha h}{w} < 1$ and $\frac{\beta g}{w} < 1$.

(e) $h = 7.1$, $g = 1.0$; $E = (0.2619, 0.8056)$; $\frac{\alpha h}{w} \geq 1$ and $\frac{\beta g}{w} < 1$.

Fig. 1 Best reply functions $R_P(e)$ and $R_A(a)$ as defined by Eqs. (9) and (10), $\overline{R}_P(e)$ and $\overline{R}_A(a)$ as defined by Eqs. (12) and (13), interior Nash equilibrium E and border equilibrium $\overline{E}$, when $\alpha, \beta = 3$, and $w = 7.46$

Similarly, when $\frac{\beta g}{w} < 1$, depending on the other player's strategy, equilibrium e^* might not be interior to the interval [0, 1] and, in order to be feasible, the constraint is binding. In this case, the agent's best reply function is:

$$\overline{R}_A\,(a) = \begin{cases} \left(\frac{aw}{g\beta}\right)^{\frac{1}{\beta-1}} & \text{if } 0 \le a \le \frac{\beta g}{w} \\ 1 & \text{if } \frac{\beta g}{w} < a \le 1. \end{cases} \tag{13}$$

Both in (12) and (13) the constant branch vanishes respectively when $\alpha h/w > 1$ and $\beta g/w > 1$ as their defining intervals become empty. However, not all the cases defined by (12) and (13) may occur. In fact, the agent's equilibrium defined by $e^* = \overline{R}_A\,(a) = 1$ is not sustainable, as it would imply that the principal would respond by not inspecting since it would no longer be optimal and so would $e = 1$. Vice versa, the principal's equilibrium defined by $a^* = \overline{R}_P\,(e) = 1$ is sustainable because by (11) the agent's response is $e = \left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}}$ that is smaller than $1 - \frac{\alpha h}{w}$; therefore, by (12) it is $a^* = 1$, and the sustainable equilibrium is $(a^*, e^*) = \left(1, \left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}}\right)$. The possible cases are represented in Fig. 1; the interior Nash equilibrium is denoted by E and by $\overline{E}$ when it is on the border.

- When $(\alpha, \beta) = (1, 1)$ the payoff functions (3) are both linear:

$$\pi_P\,(a, e) = ve - w - ha + wa\,(1 - e) \tag{14a}$$
$$\pi_A\,(a, e) = w - ge - wa\,(1 - e) \tag{14b}$$

The first order conditions yield:

$$\begin{aligned} \frac{\partial \pi_P}{\partial a} = -h + w\,(1 - e) = 0 &\Leftrightarrow e = 1 - \frac{h}{w} \\ \frac{\partial \pi_A}{\partial e} = -g + wa = 0 &\Leftrightarrow a = \frac{g}{w} \end{aligned} \tag{15}$$

Therefore, when the inspection accuracy is $a = g/w$ and the effort level is $e = 1 - h/w$, both players have no individual incentive to deviate, and $(a^*, e^*) = (g/w, 1 - h/w)$ is the unique equilibrium. These levels are respectively the probabilities of inspecting and working in the mixed-strategy equilibrium (1) of the standard inspection game Γ^0. This can be explained because, as noted by Fudenberg and Tirole (1991), player i's payoff to a mixed-strategy profile is a linear function of player i's mixing probability.

- Cases $(\alpha, \beta) \ge (1, 1)$ with $(\alpha, \beta) \not> (1, 1)$ follow from the considerations made above. For instance, assume that $\alpha = 1$, that is, the principal's payoff is linear. When $\frac{\beta g}{w}\left(1 - \frac{h}{w}\right)^{\beta-1} < 1$, the unique Nash equilibrium is the solution of the system:

$$\begin{cases} -h + w\,(1 - e) = 0 \\ R_A\,(a) = \left(\frac{aw}{\beta g}\right)^{\frac{1}{\beta-1}} = e \end{cases} \tag{16}$$

that is, $(a^*, e^*) = \left(\frac{\beta g}{w}\left(1 - \frac{h}{w}\right)^{\beta-1}, 1 - \frac{h}{w}\right)$; the special case $(a^*, e^*) = \left(\frac{\beta g}{w}, 1\right)$ occurs when $h = 0$, that is, when the inspection cost is zero. The same reasoning applies in the symmetric case with $\beta = 1$: when $1 - \frac{h\alpha}{w}\left(\frac{g}{w}\right)^{\alpha-1} > 0$, the unique equilibrium is $(a^*, e^*) = \left(\frac{g}{w}, 1 - \frac{h\alpha}{w}\left(\frac{g}{w}\right)^{\alpha-1}\right)$; similarly, the special case $(a^*, e^*) = \left(1, 1 - \frac{h\alpha}{w}\right)$ occurs when $g = w$, that is, when the wage equals the cost of effort. However, both special cases are ruled out by assumptions on costs. $\qquad\square$

Proposition 4 *When $(\alpha, \beta) > (1, 1)$ a sufficient condition to have an interior solution of* (11), *i.e.,* $E \in (0, 1)^2$ *is*

$$\alpha h > w \tag{17}$$

Proof Let us consider the principal's best reply (9), which is decreasing in e and continuous. Let us consider its inverse function and introduce the auxiliary function

$$F(a) := \left(1 - \frac{\alpha h a^{\alpha-1}}{w}\right) - R_A(a) = \left(1 - \frac{\alpha h a^{\alpha-1}}{w}\right) - \left(\frac{wa}{\beta g}\right)^{\frac{1}{\beta-1}} \tag{18}$$

Function F is continuous, decreasing w.r.t. a, and $F(0) = 1$. As $R_A(a)$ is increasing w.r.t. a, a sufficient condition to have $F(1) < 0$ is

$$\left(1 - \frac{\alpha h 1^{\alpha-1}}{w}\right) < 0 \tag{19}$$

which holds if and only if (17) holds. In this case function F has a zero in $[0, 1]$ and the Nash equilibrium is interior. □

A few examples are illustrated in Fig. 1: in Fig. 1a condition (17) holds and the Nash equilibrium is interior; however, in Figs. 1c and 1d it is still interior to $U = [0, 1]^2$ as condition (17) is not necessary.

4.3 Case: $\alpha < 1$ and $\beta > 1$ or $\alpha > 1$ and $\beta < 1$, that is, $(\alpha - 1)(\beta - 1) < 0$

Proposition 5 *In the inspection game Γ with continuous strategies, with $\alpha < 1$ and $\beta > 1$, the Nash equilibrium in pure strategies exists and is unique if and only if*

$$\left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}} < 1 - \frac{h}{w}. \tag{20}$$

In such a case, the equilibrium is $(a^*, e^*) = \left(1, \left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}}\right)$.

Proof This case follows from Proposition 2 and the above discussion. When $\alpha < 1$ either $a = 0$ or $a = 1$ are best replies. As we have seen above, we cannot have an equilibrium with $a^* = 0$, therefore, we are left with $a^* = 1$ which excludes $e^* = 0$ and $e^* = 1$.

From (3a) we have that $a^* = 1$ is optimal for any $e^* \in (0, 1)$ if and only if

$$\pi_P(0, e^*) < \pi_P(1, e^*)$$

that is

$$e^* < 1 - \frac{h}{w}. \tag{21}$$

Furthermore, $e^* \in (0, 1)$ must be the best reply to $a^* = 1$, that is

$$\left(\frac{1w}{g\beta}\right)^{\frac{1}{\beta-1}} < 1$$

as, otherwise principal's accuracy would drop to 0. As $\beta > 1$, we have

$$\left(\frac{1}{g}\frac{w}{\beta}\right)^{\frac{1}{\beta-1}} < 1 \iff \frac{g\beta}{w} > 1.$$

Therefore, from (13) we have

$$e^* = \left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}} \tag{22}$$

Putting together (21) and (22) we obtain (20). Finally, as we assumed $h < w$, (20) implies (21) and $(a^*, e^*) = \left(1, \left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}}\right)$. □

In this equilibrium the inspection accuracy is full and for the agent the marginal cost of effort is equal to the marginal cost of punishment.

Similarly:

Proposition 6 *In the inspection game Γ with continuous strategies, with $\alpha > 1$ and $\beta < 1$, the Nash equilibrium in pure strategies exists and is unique if and only if*

$$\left(\frac{w}{h\alpha}\right)^{\frac{1}{\alpha-1}} < \frac{g}{w}. \tag{23}$$

Then the equilibrium is $(a^*, e^*) = \left(\left(\frac{w}{h\alpha}\right)^{\frac{1}{\alpha-1}}, 0\right)$.

Proof The proof is analogous to the proof of Proposition 5 and therefore is omitted. □

In this equilibrium the effort is null and for the principal the marginal cost of accuracy is equal to the marginal side payment from salary cut.

4.4 Existence and uniqueness of the Nash equilibrium of game Γ

On the whole, our analysis can now be summarized in the scheme in Fig. 2 as follows:

Theorem 1 *In the continuous version of the inspection game, where the inspecting cost of the principal is $c_P(a) = ha^\alpha$ and the agent bears a cost of fatigue $c_A(e) = ge^\beta$:*

- *if $(0, 0) < (\alpha, \beta) \leq (1, 1)$ then the Nash equilibrium does not exist;*
- *if $(\alpha, \beta) > (1, 1)$ then the Nash equilibrium always exists and is unique;*
- *otherwise, the Nash equilibrium when exists is unique.*

Proof The proof follows from Propositions 2, 3, 5, and 6. □

When modeling organizational interactions, the trade-off between analytical tractability and extensive simplification is well known (Ethiraj and Levinthal 2009; Dal Forno and Merlone 2010); in particular, with regard to the principal-agent theory, it is difficult to derive analytical solutions (Zhou 2002). When condition (17) holds, in some special cases the unique feasible solution of system (11) can be found analytically, for example:

- $\alpha = \beta = 2$

$$(a^*, e^*) = \left(\frac{2gw}{4gh + w^2}, \frac{w^2}{4gh + w^2}\right); \tag{24}$$
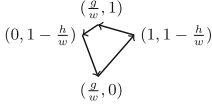
|	$\alpha < 1$	$\alpha = 1$	$\alpha > 1$
$\beta > 1$	Proposition 5 if $\left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}} < 1 - \frac{h}{w}$ $\overline{E} = \left(1, \left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}}\right)$ otherwise, $\not\exists$ NE	Proposition 3 if $\frac{\beta g}{w}\left(1 - \frac{h}{w}\right)^{\beta-1} < 1$ $E = \left(\frac{g\beta}{w}\left(1 - \frac{h}{w}\right)^{\beta-1}, 1 - \frac{h}{w}\right)$ otherwise, $\not\exists$ NE	Proposition 3 Either E as solution of the system: $\begin{cases} a^* = \left(\frac{w(1-e^*)}{\alpha h}\right)^{\frac{1}{\alpha-1}} \\ e^* = \left(\frac{wa^*}{\beta g}\right)^{\frac{1}{\beta-1}} \end{cases}$ or $\overline{E} = \left(1, \left(\frac{w}{g\beta}\right)^{\frac{1}{\beta-1}}\right)$
$\beta = 1$	Proposition 2 $\not\exists$ NE	Proposition 3 $E = \left(\frac{g}{w}, 1 - \frac{h}{w}\right)$	Proposition 3 if $1 - \frac{h\alpha}{w}\left(\frac{g}{w}\right)^{\alpha-1} > 0$ $E = \left(\frac{g}{w}, 1 - \frac{h\alpha}{w}\left(\frac{g}{w}\right)^{\alpha-1}\right)$ otherwise, $\not\exists$ NE
$\beta < 1$	Proposition 2 $\not\exists$ NE	Proposition 2 $\not\exists$ NE	Proposition 6 if $\left(\frac{w}{h\alpha}\right)^{\frac{1}{\alpha-1}} < 1 - \frac{g}{w}$ $\overline{E} = \left(\left(\frac{w}{h\alpha}\right)^{\frac{1}{\alpha-1}}, 0\right)$ otherwise, $\not\exists$ NE

Fig. 2 Scheme of the analysis of the existence of the Nash equilibrium of game Γ with continuous strategies

- $\alpha = 2$ and $\beta = 3$

$$(a^*, e^*) = \left(\frac{w^3 - w^2\sqrt{24gh + w^2} + 12wgh}{24gh^2}, \frac{w\sqrt{24gh + w^2} - w^2}{12gh}\right); \quad (25)$$

- $\alpha = 3$ and $\beta = 2$

$$(a^*, e^*) = \left(\frac{\sqrt{48g^2hw + w^4} - w^2}{12gh}, w\frac{\sqrt{48g^2hw + w^4} - w^2}{24g^2h}\right); \quad (26)$$

- $\alpha = 3, \beta = 3$

$$(a^*, e^*) = \left(\frac{r_2 - 3r_1}{6}, \sqrt{\frac{r_2 - 3r_1}{6}\frac{w}{3g}}\right) \quad (27)$$

where

- $r_1 = \sqrt{\dfrac{8w}{9h} - \dfrac{r_3}{9\sqrt[3]{2}gh^2} - \dfrac{16\sqrt[3]{2}gw^2}{9r_3} + \dfrac{2w^3}{9\,gh^2r_2}}$

○ $r_2 = \sqrt{\frac{r_3}{\sqrt[3]{2}gh^2} + \frac{16\sqrt[3]{2}gw^2}{r_3} + \frac{4w}{h}}$

○ $r_3 = \sqrt[3]{128g^3h^3w^3 + gh^2w^6 + r_4}$

○ $r_4 = \sqrt{256g^4h^5w^9 + g^2h^4w^{12}}$.

5 The continuous inspection game with quadratic costs: $\alpha = \beta = 2$

According to Greer (2022) quadratic costs conform to the properties of economic theory and offer a better alternative to other cost functions (see also Röller 1988); therefore, as in other principal-agent contributions (Holmstrom and Milgrom 1987; Schättler and Sung 1997; Encinosa et al. 2007), from here on we will consider quadratic costs. As a consequence, (9), (10), (12) and (13) become respectively:

- when $\frac{2h}{w} \geq 1$:

$$R_P(e) = \frac{w}{2h}(1 - e) \tag{28}$$

- when $\frac{2g}{w} \geq 1$:

$$R_A(a) = \frac{aw}{2g} \tag{29}$$

- when $\frac{2h}{w} < 1$:

$$R_P(e) = \begin{cases} 1 & \text{if } 0 \leq e \leq 1 - \frac{2h}{w} \\ \frac{w}{2h}(1 - e) & \text{if } 1 - \frac{2h}{w} < e \leq 1 \end{cases} \tag{30}$$

- when $\frac{2g}{w} < 1$:

$$R_A(a) = \begin{cases} \frac{aw}{2g} & \text{if } 0 \leq a \leq \frac{2g}{w} \\ 1 & \text{if } \frac{2g}{w} < a \leq 1. \end{cases} \tag{31}$$

From Proposition 3 it follows that there exists a unique Nash equilibrium for the continuous strategies inspection game given by the intersection of the best replies. However, in the quadratic costs case, it is possible to find more stringent conditions for the existence of an interior Nash equilibrium. In particular, we have:

Proposition 7 *The Nash equilibrium $E \in \left[0, \frac{2g}{w}\right] \times \left[\left(1 - \frac{2h}{w}\right), 1\right] \subseteq U$ if and only if $\frac{2h}{w} < 1$ and $\frac{2g}{w} < 1$.*

Proof Let us consider the second branch in (30), it intersects the vertical line $a = 1$ at $e = 1 - \frac{2h}{w}$ that is the solution of $\frac{w}{2h}(1 - e) = 1$ and such an intersection is feasible if and only if $\frac{2h}{w} < 1$.

Similarly, consider the first branch in (31), its intersection with the horizontal line $e = 1$ occurs at $a = \frac{2g}{w}$ and the latter is feasible if and only if $\frac{2g}{w} < 1$. □

When Proposition 7 holds, we have

$$(a^*, e^*) = E\left(\frac{2gw}{4gh + w^2}, \frac{w^2}{4gh + w^2}\right), \tag{32}$$

as depicted in Fig. 3d. However, when Proposition 7 does not hold, the equilibrium is not necessarily interior since, due to the boundaries of set U, it might be at the borders as point

$$\left(\overline{a}^*, \overline{e}^*\right) = \overline{E}\left(1, \frac{w}{2g}\right) \tag{33}$$

in Fig. 3b. As $R_P(e)$ is piecewise linear, we cannot have a border equilibrium with $e = 1$ since we assumed a strictly positive inspection cost h. Likewise, as also the cost of fatigue g is strictly positive, the intersection of the two best reply functions, when not interior to set U, can only be on the boundary $a = 1$.

A necessary and sufficient condition for a border equilibrium $\overline{E}$ is proved in the following

Proposition 8 *When considering best reply functions* (30) *and* (29), *the Nash equilibrium is on the boundary* $a = 1$ *of the set* $U = [0, 1]^2$ *if and only if*

$$g \geq \frac{w^2}{2(w - 2h)} \tag{34}$$

Proof It is immediate to compute the coordinate of the variable points $\overline{P}$ and $\overline{A}$ where $R_P(e)$ and $R_A(a)$ intersect the border of U. In particular we have: $\overline{P}\left(1, 1 - \frac{2h}{w}\right)$; $\overline{A}\left(\frac{2g}{w}, 1\right)$ when $g < w/2$ (see Fig. 3d) and $\overline{A}\left(1, \frac{w}{2g}\right)$ when $g \geq w/2$ (see Figs. 3b and 3c). As it can be seen in Fig. 3, in order to have the border equilibrium $\overline{E} \equiv \overline{A}$, the first component of $\overline{A}$ needs to be equal to one, i.e., $g \geq w/2$, and its second component must be smaller or equal to the second component of point $\overline{P}$, that is:

$$\frac{w}{2g} \leq 1 - \frac{2h}{w} \tag{35}$$

which can be rewritten as

$$g \geq \frac{w^2}{2(w - 2h)}. \tag{36}$$

Now comparing the previous inequality to $g \geq w/2$, since
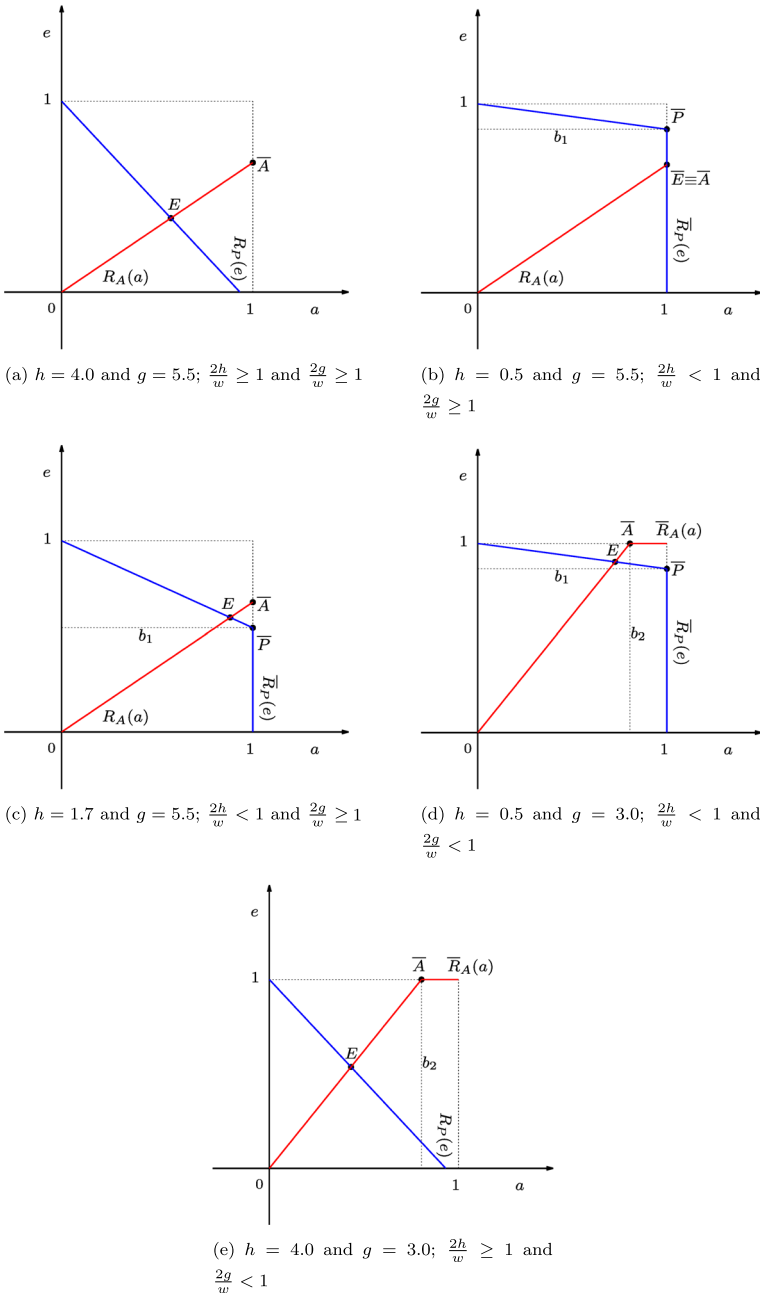
$$\frac{w}{2} \leq \frac{w^2}{2(w - 2h)}, \tag{37}$$

the most restrictive must hold and the claim is proven. □

Remark 1 It is worth to observe that, when $E \equiv \overline{E}$, by respectively imposing $a^* = \overline{a}^* = 1$ and $e^* = \overline{e}^* = w/2g$ we obtain $h = w\left(\frac{1}{2} - \frac{w}{4g}\right)$ and $g = \frac{w^2}{2(w-2h)}$. These equations will be used in the next section and can be obtained from (35) written as an equality and solved respectively for h and g.

6 Dynamic model of the inspection game with quadratic costs

Given the best reply functions (28), (29), (30) and (31) both principal and agent might ignore them and exhibit inertia in decision making (Alós-Ferrer et al. 2016) or rather, they could jump to their new optimum. The continuum between these extreme behaviors can be modeled assuming an adjustment in the direction of the optimum as it is suggested in Puu (1991);

(a) $h = 4.0$ and $g = 5.5$; $\frac{2h}{w} \geq 1$ and $\frac{2g}{w} \geq 1$

(b) $h = 0.5$ and $g = 5.5$; $\frac{2h}{w} < 1$ and $\frac{2g}{w} \geq 1$

(c) $h = 1.7$ and $g = 5.5$; $\frac{2h}{w} < 1$ and $\frac{2g}{w} \geq 1$

(d) $h = 0.5$ and $g = 3.0$; $\frac{2h}{w} < 1$ and $\frac{2g}{w} < 1$

(e) $h = 4.0$ and $g = 3.0$; $\frac{2h}{w} \geq 1$ and $\frac{2g}{w} < 1$

Fig. 3 Best reply functions R_P (e) and R_A (a) with quadratic costs as defined by Eqs. (28) and (29), $\overline{R}_P$ (e) and $\overline{R}_A$ (a) as defined by Eqs. (30) and (31); interior Nash equilibrium E and border equilibrium $\overline{E}$ as defined by Eqs. (32) and (33) respectively; $w = 7.46$

for a discussion about adjustments toward best responses in continuous and discrete time models see Matsumoto and Szidarovszky (2015). In the following we adopt this approach as it allows us to model all the behaviors between the aforementioned extremes. Let K_P and K_A denote the respective speed of adjustment of the principal and the agent; then this adjustment process can be described by the first order difference equations:

$$\begin{cases} a_{t+1} = a_t + K_P \left(R_P \left(e_t \right) - a_t \right) \\ e_{t+1} = e_t + K_A \left(R_A \left(a_t \right) - e_t \right) \end{cases} \tag{38}$$

As we have discussed in Sect. 5, depending on the different conditions on the parameter values the best reply of each player may have a different definition. A proper dynamical analysis cannot ignore these differences as they lead to a Nash equilibrium that can either be interior or at the border of the set $U = [0, 1]^2$. Furthermore, due to the presence of the boundaries in the best reply functions (30) and (31), the adjustment process (38) has different definitions depending on which region of the phase space U the state variables (a, e) belong to. The borders of these regions (when inside space U) are defined as:

$$\begin{aligned} b_1 &: e = 1 - \frac{2h}{w}, \\ b_2 &: a = \frac{2g}{w}. \end{aligned}$$

Notice that line b_1 is horizontal and line b_2 is vertical in the plane (a, e) as it can be seen in Fig. 3d. We define sets:

$$\begin{aligned} U_e &= \{(a, e) \in U : a \in [0, 1], 0 \le e \le b_1\} \\ U_a &= \{(a, e) \in U : 0 \le a \le b_2, e \in [0, 1]\} \\ \overline{U}_e &= \{(a, e) \in U : a \in [0, 1], b_1 < e \le 1\} \\ \overline{U}_a &= \{(a, e) \in U : b_2 < a \le 1, e \in [0, 1]\} \end{aligned}$$

From the game perspective, in $\overline{U}_a$ the principal's inspection level is larger than in U_a, while in set $\overline{U}_e$ the effort exerted by the agent is larger than in U_e.

The dynamical system (38) can be written as a continuous two-dimensional piecewise smooth map $T : U \to U$, with $U = [0, 1]^2$ as both $R_P : [0, 1] \to [0, 1]$ and $R_A : [0, 1] \to [0, 1]$; map T is defined by different functions on the partitions of set U. More precisely, map T is a piecewise linear map, hence the local bifurcation of a fixed point only depends on the corresponding linear map, but we will see that the global behavior in the phase space depends on the interaction between the linear maps defined also in the other regions of U. For this reason, even if map T has no fixed points in the other regions as they are all virtual, the eigenvalues of these virtual fixed points are important to the global behavior of map T.

For the sake of clarity, we distinguish the four maps arising from the different definitions of the best replies: (28), (29), (30), and (31).

6.1 Parameter conditions: $\frac{2h}{w} \ge 1$, $\frac{2g}{w} \ge 1$

With these parameter values the borders of the regions dividing set U are outside U, as $b_1 < 0$ and $b_2 > 1$; therefore, the best reply functions are (28) and (29), and map T_1 is defined as:

$$T_1 : \begin{cases} a_{t+1} = (1 - K_P) a_t - \dfrac{w}{2h} K_P e_t + \dfrac{w}{2h} K_P \\ e_{t+1} = \dfrac{w}{2g} K_A a_t + (1 - K_A) e_t \end{cases} \tag{39}$$

The unique interior fixed point of map T_1 is $E = \left(\frac{2gw}{4gh+w^2}, \frac{w^2}{4gh+w^2} \right)$ as illustrated in Fig. 3a. The Jacobian matrix is

$$J_1 = \begin{pmatrix} 1 - K_P & -\dfrac{w}{2h} K_P \\ \dfrac{w}{2g} K_A & 1 - K_A \end{pmatrix} \tag{40}$$

with

$$\text{Tr}\,(J_1) = 2 - K_P - K_A,$$

$$\det\,(J_1) = (1 - K_P)(1 - K_A) + \frac{w^2}{4gh} K_P K_A = 1 - K_P - K_A + \left(1 + \frac{w^2}{4gh} \right) K_P K_A$$

and characteristic equation

$$\lambda^2 + (K_P + K_A - 2)\lambda + 1 - K_P - K_A + \left(1 + \frac{w^2}{4gh} \right) K_P K_A = 0$$

A necessary and sufficient condition for the characteristic equation to have its roots inside the unit circle is given by the following three conditions:

$$1 - \text{Tr}\,(J_1) + \det\,(J_1) > 0; \quad 1 + \text{Tr}\,(J_1) + \det\,(J_1) > 0; \quad 1 - \det\,(J_1) > 0; \tag{41}$$

or, equivalently:

$$\begin{cases} \left(1 + \dfrac{w^2}{4gh} \right) K_P K_A > 0 \\[3mm] 4 - 2K_P - 2K_A + \left(1 + \dfrac{w^2}{4gh} \right) K_P K_A > 0 \\[3mm] K_P + K_A - \left(1 + \dfrac{w^2}{4gh} \right) K_P K_A > 0 \end{cases} \tag{42}$$

The first condition is true for all $K_P, K_A \in (0, 1]$. When either one of the two parameters is zero, say $K_A = 0$, the second function of (39) becomes $e_{t+1} = e_t$ for all $t \geq 0$. That is, for any given initial condition e_0 the agent will keep choose the same level e_0 of effort indefinitely. Consequently, the principal will adapt the level of inspection given by the best reply to e_0 and the system is in equilibrium. In other words, when $K_A = 0$ there is a continuum of stable fixed points $(R_P(e_0), e_0)$ for any initial condition $e_0 \in [0, 1]$. The same reasoning also applies to $K_P = 0$. The second condition of (42) is always satisfied due to the positivity of the parameters and the fact that K_P and K_A are not greater than one. The last condition is always satisfied too, due to the additional conditions on parameters h, g and w.

We have just proved the following proposition about the stability of the interior equilibrium E of map T_1:

Proposition 9 *When considering the dynamic inspection game* (39) *with quadratic costs, parameter conditions* $\frac{2h}{w} \geq 1$, $\frac{2g}{w} \geq 1$, *and best reply functions* (28) *and* (29), *the unique interior Nash equilibrium E is always globally stable.*

6.2 Parameter conditions: $\frac{2h}{w} \geq 1$, $\frac{2g}{w} < 1$

The best reply functions to be considered are, respectively, (28) and (31); the dynamical system (38) becomes:

$$T_2: \begin{cases} \begin{cases} a_{t+1} = (1 - K_P)\,a_t - \dfrac{w}{2h}K_P e_t + \dfrac{w}{2h}K_P \\ e_{t+1} = \dfrac{w}{2g}K_A a_t + (1 - K_A)\,e_t \end{cases} & \text{if } (a_t, e_t) \in U_a \cap \overline{U}_e \\[6pt] \begin{cases} a_{t+1} = (1 - K_P)\,a_t - \dfrac{w}{2h}K_P e_t + \dfrac{w}{2h}K_P \\ e_{t+1} = (1 - K_A)\,e_t + K_A \end{cases} & \text{if } (a_t, e_t) \in \overline{U}_a \cap \overline{U}_e \end{cases} \tag{43}$$

By the definition of map T_2 and conditions on parameters $\frac{2h}{w} \geq 1$, $\frac{2g}{w} < 1$, only two regions of the phase space U are involved in the asymptotic dynamics, that is, $U_a \cap \overline{U}_e$ and $\overline{U}_a \cap \overline{U}_e$, as illustrated in Fig. 3e. The fixed points are, respectively, $E := E_{U_a \cap \overline{U}_e}\left(\frac{2gw}{4\,gh+w^2}, \frac{w^2}{4\,gh+w^2}\right)$ for the function defined in $U_a \cap \overline{U}_e$, and $\overline{E}' := E_{\overline{U}_a \cap \overline{U}_e}(0, 1)$ for the function defined in $\overline{U}_a \cap \overline{U}_e$. However, only the interior point E is real while the corner point $\overline{E}'$ is virtual as it lies outside its proper partition $\overline{U}_a \cap \overline{U}_e$. This virtual point is the Pareto efficient solution of the static game, that is neither sustainable in Γ^0 nor in Γ.

The Jacobian matrix of the branch of map T_2 defined in region $U_a \cap \overline{U}_e$ is the same of (40), therefore, the interior fixed point E is stable. On the other hand, the Jacobian matrix of the branch defined in region $\overline{U}_a \cap \overline{U}_e$ is

$$J_2 = \begin{pmatrix} 1 - K_P & -\dfrac{w}{2h}K_P \\ 0 & 1 - K_A \end{pmatrix}. \tag{44}$$

The eigenvalues of J_2 are $\lambda_1 = 1 - K_P$ and $\lambda_2 = 1 - K_A$; both are real, positive and less than one, so the virtual corner equilibrium $\overline{E}'$ is attracting in the whole parameter space (K_P, K_A). This implies that any initial condition belonging to the interior of region $\overline{U}_a \cap \overline{U}_e$ is mapped toward the virtual attractor $\overline{E}'$, and is forced to enter region $U_a \cap \overline{U}_e$ where it will be attracted toward equilibrium E.

We have just proved the following proposition about the stability of the interior Nash equilibrium:

Proposition 10 *When considering the dynamic inspection game* (43), *with quadratic costs, parameter conditions* $\frac{2h}{w} \geq 1$, $\frac{2g}{w} < 1$, *and best reply functions* (28) *and* (31), *the unique interior Nash equilibrium E is globally stable.*

6.3 Parameter conditions: $\frac{2h}{w} < 1$, $\frac{2g}{w} \geq 1$

In this case the best reply functions are respectively (30) and (29) and the map is defined as:

$$T_3: \begin{cases} \begin{cases} a_{t+1} = (1 - K_P)\,a_t + K_P \\ e_{t+1} = \dfrac{w}{2g}K_A a_t + (1 - K_A)\,e_t \end{cases} & \text{if } (a_t, e_t) \in U_a \cap U_e \\[6pt] \begin{cases} a_{t+1} = (1 - K_P)\,a_t - \dfrac{w}{2h}K_P e_t + \dfrac{w}{2h}K_P \\ e_{t+1} = \dfrac{w}{2g}K_A a_t + (1 - K_A)\,e_t \end{cases} & \text{if } (a_t, e_t) \in U_a \cap \overline{U}_e \end{cases} \tag{45}$$

Also in this case, by the definition of map T_3 and parameter conditions $\frac{2h}{w} < 1$, $\frac{2g}{w} \geq 1$, the phase space U is partitioned in only two regions, that is, $U_a \cap U_e$ and $U_a \cap \overline{U}_e$ (see Figs. 3b and 3c). The fixed points are, respectively, $\overline{E} := \overline{E}_{U_a \cap U_e} \left(1, \frac{w}{2g}\right)$ for the function defined in region $U_a \cap U_e$, and $E := \overline{E}_{U_a \cap \overline{U}_e} \left(\frac{2gw}{4gh+w^2}, \frac{w^2}{4gh+w^2}\right)$ for the function defined in region $U_a \cap \overline{U}_e$. However, they never occur simultaneously: when the necessary condition (34) holds, then E is virtual while $\overline{E}$ is real; vice versa, when the necessary condition (34) does not hold, then E is real while $\overline{E}$ is virtual. In fact, if $\overline{E}$ is real, then it belongs to partition $U_a \cap U_e$ (that is, it lies below line b_1) and, therefore, its second component must satisfy the condition $w/2g < 1 - 2h/w$. On the other hand, the second component of E is $\frac{w^2}{4gh+w^2}$ and, by inequality (34) and simple algebra, we obtain that $\frac{w^2}{4gh+w^2} < 1 - 2h/w$ (that is, E lies below line b_1 too) and therefore E belongs to partition $U_a \cap U_e$ instead of belonging to its proper partition $U_a \cap \overline{U}_e$; therefore, E is virtual. The same reasoning can be applied to prove that if $\overline{E}$ is virtual (that is, it lies above line b_1), then condition (34) does not hold, and E lies above line b_1 too; therefore, E is real.

The Jacobian matrix of the branch of map T_3 defined in region $U_a \cap U_e$ is

$$J_3 = \begin{pmatrix} 1 - K_P & 0 \\ \dfrac{w}{2g} K_A & 1 - K_A \end{pmatrix}. \tag{46}$$

Also in this case the eigenvalues are $\lambda_1 = 1 - K_P$ and $\lambda_2 = 1 - K_A$, and both are real, positive, and less than one, so the border equilibrium $\overline{E}$ is attracting in the whole parameter space (K_P, K_A).

The second branch of map T_3, instead, has the same Jacobian matrix (40) of map T_1 and, therefore, all the stability conditions in (42) hold for all $K_P, K_A \in [0, 1]$, but the last one. In fact, with parameter condition $\frac{2h}{w} < 1$ and $\frac{2g}{w} \geq 1$, the last condition is here satisfied if

$$\frac{K_P K_A}{K_P + K_A} < \frac{4gh}{4gh + w^2}.$$

We have just proved the following proposition about the stability of the unique Nash equilibrium of the dynamic game (45):

Proposition 11 *Let us consider the dynamic inspection game* (45), *with quadratic costs, parameter conditions* $\frac{2h}{w} < 1$, $\frac{2g}{w} \geq 1$, *and best reply functions* (30) *and* (29). *Furthermore, if:*

- $g \geq \frac{w^2}{2(w-2h)}$, *then there exists a unique border Nash equilibrium $\overline{E}$ that is globally stable.*
- $g < \frac{w^2}{2(w-2h)}$, *then there exists a unique interior Nash equilibrium E which is globally asimptotically stable if*

$$\frac{K_P K_A}{K_P + K_A} < \frac{4gh}{4gh + w^2}. \tag{47}$$

The intuition underlying condition (47) can be better explained when discussing the bifurcation the equilibrium may undergo, and we defer it for the moment at the end of this section. Moreover, when $\frac{K_A K_P}{K_A + K_P} > \frac{4gh}{4gh + w^2}$, numerical evidence of the existence of cycles will be provided in the next subsection.

Finally, the border equilibrium $\overline{E}$ analyzed in Proposition 11 corresponds to the situation in which the principal's inspection accuracy is maximum ($a = 1$) and the agent's effort

$(e = w/2g)$ is the best reply to such a level of accuracy. Both the interior and the border equilibrium of the above propositions are the Nash equilibria of the inspection game Γ we discussed in the previous sections.

6.4 Parameter conditions: $\frac{2h}{w} < 1$, $\frac{2g}{w} < 1$

This is the last case to be considered, that is, the one whose parameters partition the phase space U in four proper regions (see Fig. 3d). The best reply functions are (30) and (31), and map (38) becomes:

$$
T_4 : \begin{cases}
\begin{cases}
a_{t+1} = (1 - K_P)\, a_t + K_P \\
e_{t+1} = \dfrac{w}{2g} K_A a_t + (1 - K_A)\, e_t
\end{cases} & \text{if } (a_t, e_t) \in U_a \cap U_e \\[2em]
\begin{cases}
a_{t+1} = (1 - K_P)\, a_t - \dfrac{w}{2h} K_P e_t + \dfrac{w}{2h} K_P \\
e_{t+1} = \dfrac{w}{2g} K_A a_t + (1 - K_A)\, e_t
\end{cases} & \text{if } (a_t, e_t) \in U_a \cap \overline{U}_e \\[2em]
\begin{cases}
a_{t+1} = (1 - K_P)\, a_t + K_P \\
e_{t+1} = (1 - K_A)\, e_t + K_A
\end{cases} & \text{if } (a_t, e_t) \in \overline{U}_a \cap U_e \\[2em]
\begin{cases}
a_{t+1} = (1 - K_P)\, a_t - \dfrac{w}{2h} K_P e_t + \dfrac{w}{2h} K_P \\
e_{t+1} = (1 - K_A)\, e_t + K_A
\end{cases} & \text{if } (a_t, e_t) \in \overline{U}_a \cap \overline{U}_e
\end{cases} \tag{48}
$$

Each of the four linear branches of map T_4, defined on each of the different partitions of set U, has a fixed point; they are respectively:

$$
\overline{E} := E_{U_a \cap U_e}\left(1, \frac{w}{2g}\right); \; E := E_{U_a \cap \overline{U}_e}\left(\frac{2gw}{4gh+w^2}, \frac{w^2}{4gh+w^2}\right) \in U_a \cap \overline{U}_e;
$$
$$
\overline{E}'' := E_{\overline{U}_a \cap U_e}(1, 1); \; \overline{E}' := E_{\overline{U}_a \cap \overline{U}_e}(0, 1);
$$

for each point the subscript denotes the branch determining the fixed point.

Fixed points $\overline{E}$, $\overline{E}'$ and $\overline{E}''$ are virtual, as they do not belong to the partition where the branch of the map is defined, while E is real and is the unique fixed point of map T_4. This point corresponds to the Nash equilibria (32) of the inspection game Γ. In general, corner points $\overline{E}'$ and $\overline{E}''$ are neither fixed points of the dynamic game nor equilibria of game Γ. Nevertheless, together with $\overline{E}$, they are relevant for the dynamic behavior of map T_4 because they influence the dynamics in the related partition of the phase space (see Gardini and Tikjha 2019).

By Proposition 11, the interior real fixed point E is attractive if condition (47) holds. By Proposition 11, the border virtual fixed point $\overline{E}$ also is attractive, independently of wage and costs of inspection and effort. The branch of map T_4 defined in partition $\overline{U}_a \cap U_e$ has Jacobian matrix

$$
J_4 = \begin{pmatrix} 1 - K_P & 0 \\ 0 & 1 - K_A \end{pmatrix}.
$$

This matrix has the same eigenvalues of matrix J_3 in Sect. 6.3, therefore, the virtual fixed point $\overline{E}'' = (1, 1)$ is attracting. Finally, from Sect. 6.2 we know that the corner virtual fixed point $\overline{E}' = (0, 1)$ is attracting as well. As a consequence, any initial condition in a region different from $U_a \cap \overline{U}_e$ is mapped toward its virtual attractor. For example, any initial condition in

region $U_a \cap U_e$ will be mapped into $\overline{U}_a \cap U_e$ in a finite number of iterations (as the points tend to the virtual border fixed point $\overline{E}$ wich is attracting). Once in region $\overline{U}_a \cap U_e$, the points of the trajectory will be attracted by the stable virtual corner fixed point $\overline{E}''$ (1, 1) and will be mapped in region $\overline{U}_a \cap \overline{U}_e$ in a finite number of steps. Finally, points in region $\overline{U}_a \cap \overline{U}_e$ are subject to the attraction of the virtual corner fixed point $\overline{E}'$ (0, 1) and, in a finite number of iterations, they will enter region $U_a \cap \overline{U}_e$. Here, when (47) holds, they will asymptotically converge to the real equilibrium E, or, otherwise, to a finite period cycle bounded in U. This way it holds:

Proposition 12 *When considering the dynamic inspection game* (48), *with quadratic costs, parameter conditions* $\frac{2h}{w} < 1$, $\frac{2g}{w} < 1$, *and best reply functions* (30) *and* (31), *then there exists a unique interior Nash equilibrium E that is globally asymptotically stable for all $K_P, K_A \in [0, 1]$ such that*

$$\frac{K_P K_A}{K_P + K_A} < \frac{4gh}{4gh + w^2}. \tag{49}$$

6.5 Some remarks about the bifurcations

Since a detailed and complete bifurcation analysis is beyond the purpose of the paper, we defer it to further research. For this reason, in this paper we limit ourselves to a preliminary presentation of the bifurcation phenomena that can be observed as far as they are of economic interest in the inspection game.

By Proposition 12, for the range values of the parameters defining map T_4, we have proved that the interior equilibrium E is attracting when $\frac{K_A K_P}{K_A + K_P} < \frac{4gh}{4gh + w^2}$. It becomes repelling when $\frac{K_A K_P}{K_A + K_P} > \frac{4gh}{4gh + w^2}$, loosing its stability at the bifurcation value $\frac{K_A K_P}{K_A + K_P} = \frac{4gh}{4gh + w^2}$ with a pair of complex conjugate eigenvalues crossing the unit circle and a center bifurcation occurs. For a detailed discussion of the center bifurcations see for example Sushko et al. (2003), Sushko and Gardini (2006) and Sushko and Gardini (2008). This bifurcation is illustrated in Fig. 4a, where the boundary between the stability (in black) and instability regions of the interior Nash equilibrium E in the parameter space (K_P, K_A) is analytically derived from (47) as:

$$K_A = \frac{4gh K_P}{\left(4gh + w^2\right) K_P - 4gh}. \tag{50}$$

So far as the adjustment parameters are small, the inertia of agent and principal is large, and the Nash equilibrium E of the dynamic inspection game coincides with the Nash equilibrium of game Γ. However, depending on the wage w and the costs g and h, as the inertia diminishes, the dynamic inspection game structurally departs from the static game Γ and the unique stable equilibrium gives way to cyclical behaviors, even with large periods, that can be unpredictable and appear also chaotic when observed in reality (Svyantek and Brown 2000). Some numerical evidence will be provided in the next subsection.

However, this is not the only bifurcation possible, as a border collision bifurcation can also occur. In the interior of the regions in which set U is partitioned, map T is differentiable being linear, while lines b_1 and b_2 are boundaries along which the map is not. Also, across these borders the map changes definition. When a portion of an attractor comes into contact with one of these lines and then crosses it, a border collision bifurcation can occur; as a consequence, the sudden creation, destruction or qualitative change of an attractor can be observed. This is the case of map T_3, for instance, when the border equilibrium $\overline{E}$ crosses

line b_1 due to a change of the parameters. As both the adjustment speeds are in the interval $[0, 1]$, the left-hand side of (47) is in $[0, 1)$; when considering the right-hand side we have

$$\frac{\partial}{\partial h}\left(\frac{4gh}{4gh + w^2}\right) = \frac{4gw^2}{\left(4gh + w^2\right)^2} > 0, \tag{51}$$

$$\frac{\partial}{\partial g}\left(\frac{4gh}{4gh + w^2}\right) = \frac{4hw^2}{\left(4gh + w^2\right)^2} > 0, \tag{52}$$

and

$$\frac{\partial}{\partial w}\left(\frac{4gh}{4gh + w^2}\right) = \frac{-8ghw}{\left(4gh + w^2\right)^2} < 0. \tag{53}$$

Therefore, as either the cost of inspection h or the cost of effort g becomes small the system tends to lose stability; on the contrary, when the wage increases the system loses stability. This is illustrated for maps T_1, T_3 and T_4 in Figs. 5a and 6a, without losing generality, by considering $K_P = K_A = K$, that is, when both the principal and the agent have the same adjustment speed K. In this case condition (47) becomes:

$$K < \frac{8gh}{4gh + w^2}. \tag{54}$$

Boundary $f_1(K)$ and $f_2(K)$ between the stability and instability regions (center bifurcations) in Figs. 5b and 6b can be obtained from (54) and are respectively

$$h = f_1(K) = \frac{w^2 K}{4g(2 - K)}; \ g = f_2(K) = \frac{w^2 K}{4h(2 - K)}. \tag{55}$$

In Fig. 5a we can see that, for small values of the adjustment parameter K, as the cost of inspection h increases the border Nash equilibrium $\overline{E}$ (existing for parameters choice in the gray region) changes into the stable Nash equilibrium E (existing for parameters choice in the black region) without losing its stability as it crosses line b_1. On the contrary, for large enough value of the adjustment parameter, as the inspection cost increases (as along path p_2), the border Nash equilibrium $\overline{E}$ loses its stability via a border collision bifurcation because it crosses line b_1 from below and enters the region of instability becoming the interior unstable Nash equilibrium E where cyclic behaviors occur. This difference can be explained recalling (51) as the stability depends on principal's and agent's speed of adjustment.

Finally, from both Figs. 5a and 6a we can see that, even considering the restriction $h < g$ introduced to game Γ^0 in Fudenberg and Tirole (1991), there are interesting scenarios as the dynamic behavior is not always convergent to the equilibrium. In other words, we can provide numerical evidence that assuming that inspecting is less costly than exerting effort is not sufficient to have convergence to the equilibrium in the dynamic game.

6.6 Numerical examples

In this subsection we present some numerical examples obtained with maps T_1, T_3 and T_4. The results just proved can be observed in Figs. 4, 5 and 6.

In Fig. 4a the stability region (in black) is bounded by condition (49) from Proposition 12. When either K_P or K_A increases, the stable fixed point of map T_4 crosses the center bifurcation curve (50) and cycles are created. Figures 4b and 4c illustrate the bifurcation diagrams of the inspection intensity a and the effort level e, obtained with parameter values $w = 7.46$,

$h = 0.5$, $g = 3.0$ and speeds of adjustment $K_P = K_A = K \in [0, 1]$ as bifurcation parameters, along path p_1 depicted with the red arrow in Fig. 4a; this structure will be investigated in further research. At $K \simeq 0.1946$, which can be derived from (54), the unique and stable interior Nash equilibrium E loses stability and an attractor for which we are unable to find a finite period, yet with finite amplitude, appears. This bifurcation structure needs further investigation as well, as the respective period diagrams in Figs. 4d and 4e suggest a period adding structure. Comparing the bifurcation diagrams of inspection intensity in Fig. 4b and effort level in Fig. 4c we can see that they are qualitatively similar, showing how the dynamics of the two state variables are linked; therefore, in the following we will report only those relative to the inspection accuracy level.

The boundary Nash equilibrium $\overline{E}$ of map T_3 is always stable (see Proposition 11). The inertia, that is, the decreasing value of the speed of adjustment K_P and K_A, has a stabilizing effect (Fig. 5a), but at the same time the best reply functions have a stabilizing effect as well. Due to a variation of parameters h and g, the boundary stable equilibrium $\overline{E}$ crosses border b_1, becomes the interior equilibrium E, still remaining a fixed point of map T. Depending on the adjustment speed values, E may either remain stable or it may become unstable; in other words a border collision bifurcation occurs. Furthermore, in Figure 5a we can see the effect of the boundary Nash equilibrium $\overline{E}$ on the dynamics.

Analogously to (55) from condition (34) we can derive the constant line

$$\underline{h} = f_3(g) = w\left(\frac{1}{2} - \frac{w}{4g}\right). \tag{56}$$

This value can be obtained as showed in Remark 1 and is represented together with $f_1(K)$ in Fig. 5b. In particular with the parameter constellation used in Fig. 5, from (56) we obtain $\underline{h} \simeq 0.6382$ as represented in Fig. 5b. In Fig. 5c the one-dimensional bifurcation diagram along path p_1 for the intensity of inspection a is presented; the bifurcation is similar to those discussed in Figs. 4b and 4c and it occurs at $K \simeq 0.4888$, which can be found solving for K in $f_1(K) = 1$.

The appearance of high period attractors is not exclusively due to the increasing of the adjustment speeds K. Let us consider path p_2 in Fig. 5a where $K = 0.6$. When the inspection cost is low ($h = 0.5$), the equilibrium is $\overline{E} = (1, 0.8289)$, since when h is small the cost of inspection is small and therefore the principal's accuracy of inspection is high ($a = 1$). When the inspection cost crosses line $\underline{h} \simeq 0.6382$ equilibrium $\overline{E}$ moves from region $U_a \cap U_e$ to region $U_a \cap \overline{U}_e$ crossing border b_1, becomes the interior equilibrium E, and both inspection accuracy and effort level decrease. However, for the considered speed of adjustment K, this equilibrium is unstable and the convergence is not achieved. Rather, there is a sudden convergence to an attractor, which widens as the inspection cost h further increases (see Fig. 5d). Of course, with great inertia, for instance $K = 0.2$, the dynamics would remain stable. This suggests that predicting the dynamics after the border crossing is generally not an easy task, because it depends on the global properties of the map. It must be noted that neither one-dimensional bifurcation diagrams along path p_1 nor p_2 provide any suggestion about the period structure of the dynamics. However the one-dimensional bifurcation diagram along path p_3 and its period diagram presented respectively in Figs. 5e and 5f suggest a period adding structure.

Finally, in Fig. 6a we can see again how the boundary Nash equilibrium stabilizes the dynamics; in fact, besides $f_2(K)$ which separates the region of stability from the one with periodic cycles, from condition (34) we obtain the constant line

$$\overline{g} = f_4(h) = \frac{w^2}{2(w - 2h)}. \tag{57}$$

Also this value can be obtained as showed in Remark 1 and is represented together with $f_2(K)$ in Fig. 6b. In particular, with the parameter constellation used in Fig. 6, from (57) we obtain $\overline{g} \simeq 4.3074$.

In Fig. 6c the one-dimensional bifurcation diagrams along path p_1 for the intensity of inspection a is presented; the center bifurcation is similar to those discussed in Figs. 4b, 4c and 5c and it occurs at $K \simeq 0.1946$, which can be found solving for K in $f_2(K) = 3$. When considering the one-dimensional bifurcation diagram along path p_2, illustrated in Fig. 6d, we can see that, for sufficiently large adjustment speed, when the cost g of effort is low, the dynamics may exhibit cycles and the interior equilibrium E is unstable. When the effort cost crosses line $\overline{g} \simeq 4.3074$ the equilibrium E moves from region $U_a \cap \overline{U}_e$ to region $U_a \cap U_e$ crossing border b_1, and becomes the stable border equilibrium $\overline{E}$; in fact, when the cost of effort is high there is need for full inspection accuracy.

However, these figures do not suggest a period adding structure, at least for large values of the adjustment speed. This is confirmed when the one-dimensional bifurcation diagram along path p_3, presented in Fig. 6e, is considered: there we have a period incrementing structure as it can be seen in Fig. 6f. The periodicity regions issuing from point $(1, 0)$ at the edge of the parameter plane (K, g) in Fig. 6a are also affected by bistability. A numerical evidence of coexistence of attractors of map T_4 is provided in Fig. 7. A period-5 cycle (white dots) coexists with a period-6 cycle (black dots). However, this period incrementing structure and other interesting phenomena highlighted by the numerical simulations we provided deserve a thorough analysis that is beyond the scope of the paper and it is left to a future research.

7 Discussion, further research and conclusion

When considering cooperation, several other contributions (Doebeli and Knowlton 1998; Wahl and Nowak 1999a, b) have assumed that cooperative investment can continuously vary. This approach seems more realistic when considering interactions such as the inspection game.

The extension of the inspection game Γ^0 to the continuous version Γ we have proposed, when costs are convex, presents a Nash equilibrium in pure strategies and overcomes the difficulty of interpreting mixed strategies equilibria in one shot interactions.

The dynamical interaction we introduce is a step toward the modeling of temporal interactions in organizational settings as "most workplace phenomena take place in dynamic social settings and emerge over time" (Lehmann-Willenbrock and Allen 2018, p. 325).

When comparing each other Table 1, Figs. 2 and 7 it is possible to see how the lack of pure strategies equilibrium in the classic game Γ^0—which can be seen as a cycle of retaliations on the game payoff matrix—and the unique mixed strategies Nash equilibrium are affected by considering the continuous version of the game and its dynamical version. The continuous game Γ, depending on the concavity/convexity of the cost functions may modify the possible cycle of retaliation and allow for pure strategies equilibria. Finally, the introduction of the dynamic adaptation process further modifies the cycle of retaliation and coexistence emerges.

In this paper we emphasized the effects of binding reaction constraints on both the existence and stability of Nash equilibria of the continuous inspection game Γ, and the non standard bifurcation routes leading to the creation of attractors. In the first case, we have been able to provide some conditions for the global asymptotical stability of the equilibrium
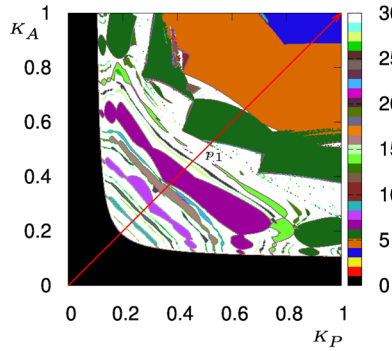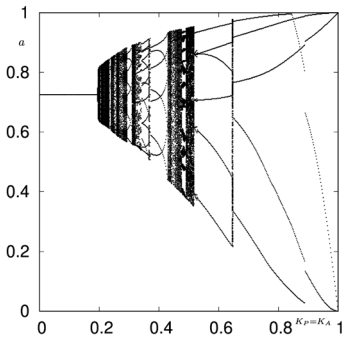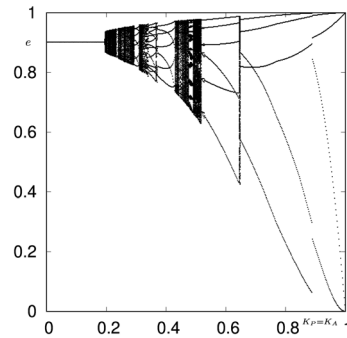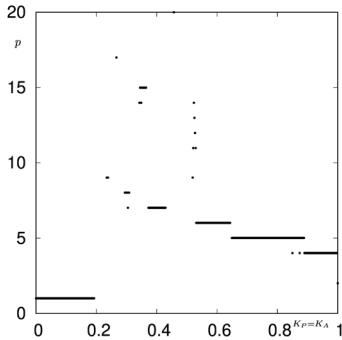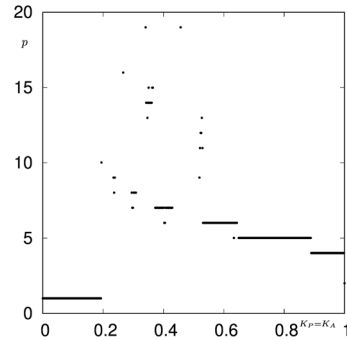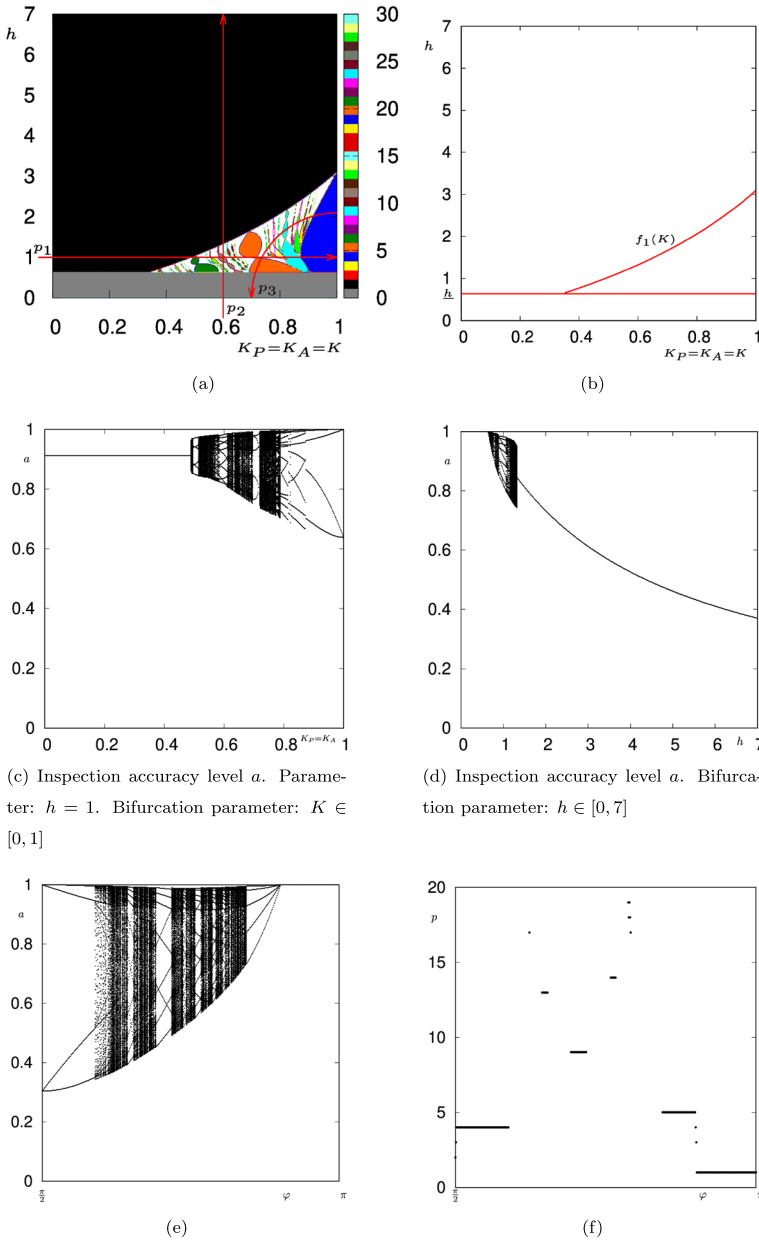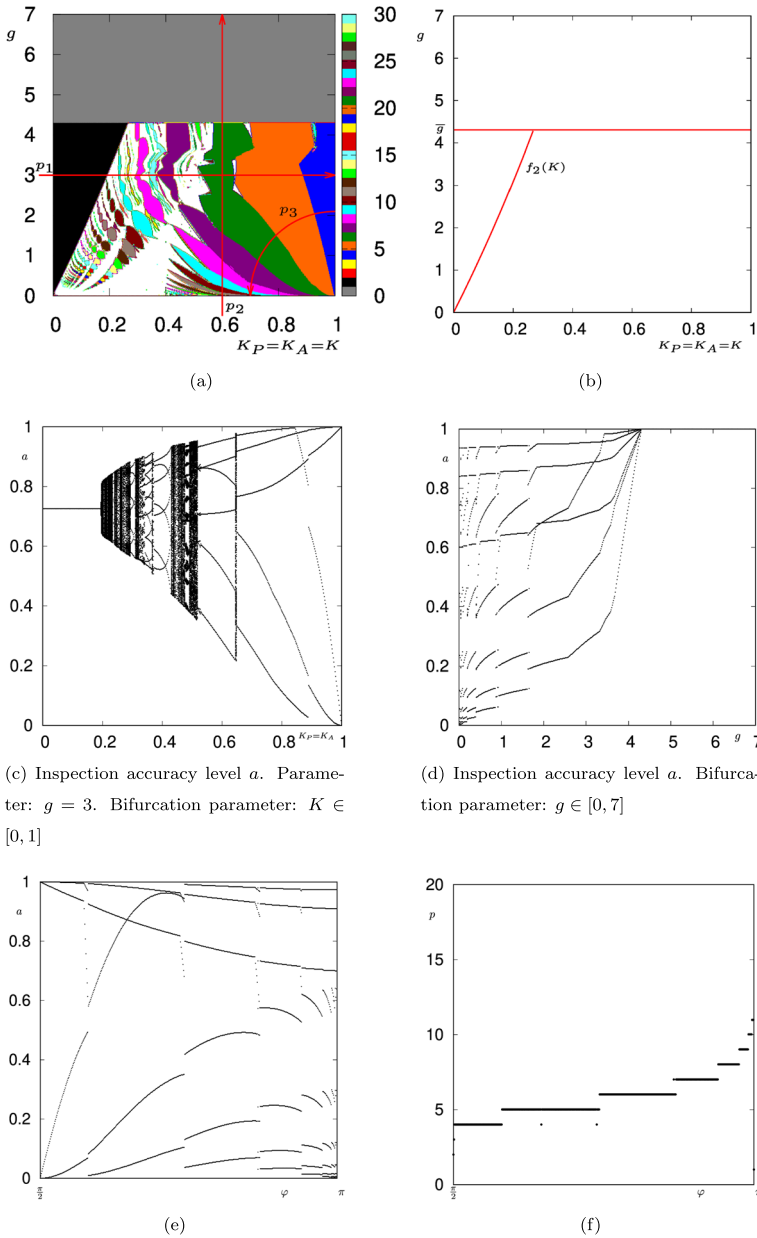
Fig. 4 Parameters: $w = 7.46$, $h = 0.5$, $g = 3.0$. (**a**) 2D bifurcation diagram with K_P and K_A as bifurcation parameters for map T_4. In the black area the set of parameter values leads to an interior fixed point. (**b**) and (**c**) One-dimensional bifurcation of the state variables along the parameter path marked p_1 in (**a**); (**d**) and (**e**) their respective period diagrams

(a)

(b)

(c) Inspection accuracy level a. Parameter: $h = 1$. Bifurcation parameter: $K \in [0, 1]$

(d) Inspection accuracy level a. Bifurcation parameter: $h \in [0, 7]$

(e)

(f)

Fig. 5 Parameters: $w = 7.46$, $g = 4.5$. (**a**) 2D bifurcation diagram with $K = K_P = K_A \in [0, 1]$ and $h \in [0, 7]$ as bifurcation parameters for map T_3 for $h < 3.73$, and map T_1 for $h \geq 3.73$. The gray area represents parameter values leading to a border fixed point. (**b**) Bifurcation curves f_1 and $\underline{h}$ derived respectively from Equation (55) and (56). One-dimensional bifurcation diagrams of the state variable a along the parameter path marked p_1 (**c**) and p_2 (**d**) in (**a**). One-dimensional bifurcation diagram of the state variable a along the parameter path marked p_3 (**e**) in (**a**) and its relative period diagram (**f**); angle φ is measured along arc p_3

(a)

(b)

(c) Inspection accuracy level a. Parameter: $g = 3$. Bifurcation parameter: $K \in [0, 1]$

(d) Inspection accuracy level a. Bifurcation parameter: $g \in [0, 7]$

(e)

(f)

Fig. 6 Parameters: $w = 7.46$, $h = 0.5$. (**a**) 2D bifurcation diagram with $K = K_P = K_A \in [0, 1]$ and $g \in [0, 7]$ as bifurcation parameters for map T_4 for $g < 3.73$, and map T_3 for $g \geq 3.73$. The gray area represents parameter values leading to a border fixed point. (**b**) Bifurcation curves f_2 and $\bar{g}$ derived respectively from Equation (55) and (57). (**c**) One-dimensional bifurcation diagram of the state variable a along the parameter path marked p_1 in (**a**). One-dimensional bifurcation diagrams of the state variable a along the parameter path marked p_2 (**d**) and p_3 (**e**) in (**a**). (**f**) period diagram relative to (**e**), angle φ is measured along arc p_3

Fig. 7 Phase space (a, e) with basins of attraction of two coexisting attractors of map T_4: parameter values $h = 0.5$, $g = 1.3888$ and $K_A = K_P = 0.8076$. The two trajectories can be obtained with initial conditions $(a_0, e_0) = (0.7159, 0.3626)$ for the period-5 cycle, and $(a_0, e_0) = (0.8131, 0.1442)$ for the period-6 cycle
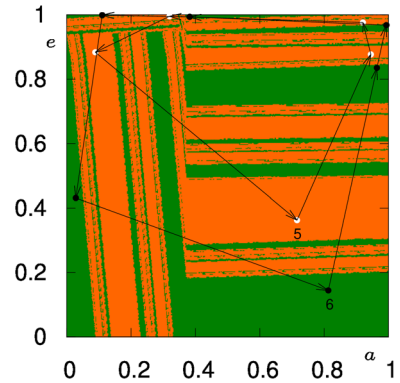
when the best reply functions are bounded. In the second case, we analyzed how the reduction of the inspection cost can induce changes on the kind of attractors characterizing the long-run dynamics. In particular, there is evidence of the emergence of a complex behavior of both the principal (in the level of inspection accuracy) and the agent (in the level of effort exerted in the task).

It is interesting to relate our findings to the extant literature in management and negotiation. The collectively optimal solution consists of the agent working ($e = 1$) and the principal not inspecting ($a = 0$); this solution is Pareto efficient yet not sustainable given the game theory assumptions. It is interesting to note that this solution would become sustainable under the assumptions of McGregor's Theory Y (McGregor 1960), however in this case the payoffs for the agent cannot be those of the standard inspection game. By contrast, in the dynamic setting with the payoffs presented in Table 1, which are consistent with McGregor's Theory X, the outcome ($a = 0, e = 1$) is an (unstable) equilibrium only if $K_A = 0$. In fact, assuming that the agent keeps working when the principal does not inspect is unrealistic: the side payment/punishment is a costly way to remind the consequences of shirking. The dynamic interaction we consider sustains collaboration by the shadow of the past and can be contrasted to some insights from Gibbons and Henderson (2012) where relational contracts are defined as "an economist's term for collaboration sustained by the shadow of the future as opposed to formal contracts enforced by courts" [p. 1350]. The reasons of this contrast lie both in the time line considered when approaching the principal agent problem in relational contracts (Levin 2003; Board 2011; Gibbons and Henderson 2012) which is different from the one in the inspection game Γ^0 (Fudenberg and Tirole 1991) and the approach used to model the interaction over time. In this sense, in the inspection game both parts are more symmetrical in creating joint value (Bridoux and Stoelhorst 2016).

The difficulty of sustaining the collective optimal solution could be resolved by a negotiation solution, following the approach outlined in Nalebuff (2020): as the contributions by the principal and the agent are equal, the pie is split between the parts. In Nalebuff (2020) the pie is defined as the total value minus what principal and agent may obtain separately. For the sake of simplicity, we assume that their participation constraint is zero: in this case the share of the pie for each is $(v - g)/2$ and $w = (v - g)/2$; in other words, they equally share costs and benefits. Finally, this solution is equivalent to the outcome of the divide-and-choose procedure (Brams and Taylor 1996).

The analysis we presented in this paper can be extended in different directions. First, different adjustment mechanisms may be considered in order to take into account other management styles. Second, the role of delays in the dynamics may be studied, especially

when considering the side payment by the principal. Finally, the resulting map is a two-dimensional piecewise linear map: these kind of maps have been studied extensively. In future work we will analyze the several interesting phenomena for which we have provided numerical evidence: among the others, the border collision bifurcation, the center bifurcation and the period incrementing structures.

Declarations

Conflict of interest The Authors have also no relevant financial or non-financial interests to disclose.

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