V–MONOTONE COCYCLES AND ALMOST PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract. In the present paper we consider a special class of equations $x' = f(t,x)$ when the function $f : \mathbb{R} \times E \to E$ (E is a finite-dimensional Banach space) is V–monotone with respect to (w.r.t.) $x \in E$, i.e. there exists a continuous non-negative function $V : E \times E \to \mathbb{R}_+$, which equals to zero only on the diagonal, so that the numerical function $\alpha(t) := V(x_1(t), x_2(t))$ is non-increasing w.r.t. $t \in \mathbb{R}_+$, where $x_1(t)$ and $x_2(t)$ are two arbitrary solutions of (1) defined and bounded on $\mathbb{R}_+$.

The main result of the paper contains the solution of the problem of V.V.Zhikov (1973): every finite-dimensional V-monotone almost periodic differential equation with bounded solutions admits at least one almost periodic solution.

1. Introduction

The problem of the almost periodicity of solutions of non-linear almost periodic ordinary differential equations

$$x' = f(t,x) \tag{1}$$

was studied by many authors (see, for example, [3, 4, 5, 6, 7, 8, 14, 20, 21] and the bibliography therein).

In the present paper we consider a special class of equations (1), where the function $f : \mathbb{R} \times E \to E$ (E is a finite-dimensional Banach space) is V–monotone with respect to (w.r.t.) $x \in E$, i.e. there exists a continuous non-negative function $V : E \times E \to \mathbb{R}_+$ which equals to zero only on the diagonal so that the numerical function $\alpha(t) := V(x_1(t), x_2(t))$ is non-increasing w.r.t. $t \in \mathbb{R}_+$, where $x_1(t)$ and $x_2(t)$ are two arbitrary solutions of (1) defined and bounded on $\mathbb{R}_+$. This class of non-linear differential equations (1) is interesting enough and well studied (see, for example, [5, 6, 14, 24] and the bibliography therein).

If the function $\alpha(t) = V(x_1(t), x_2(t))$ is strictly decreasing, then equation (1) admits a single almost periodic solution if there exists a bounded solution on $\mathbb{R}_+$.

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In general case (when the function $\alpha(t) := V(x_1(t), x_2(t))$ is non-increasing) the proof of the existence of an almost periodic solution (under the assumption that a bounded solution exists on $\mathbb{R}$) turns out to be difficult. For example, the difficulty consists in the fact that equation (1) might have an infinite number of bounded solutions on $\mathbb{R}$ (for instance, all solutions might be bounded on $\mathbb{R}$) and it is not clear how should we pick an almost periodic solution out of this set of bounded solutions.

Even for functions V of the form $V(x_1, x_2) := W(x_1 - x_2)$, where W is homogeneous and convex on E, there is no known results for the euclidean space of dimension ≥ 4 (**Problem of V.V.Zhikov** [24]).

For $dimE \leq 3$ this problem was solved by V.V.Zhikov [24] using the methods which are not related to monotonicity.

The main result of this paper states that every V–monotone almost periodic equation (1) with bounded solutions admits at least one almost periodic solution.

Let $\varphi(t, f, z)$ be a unique solution of V-monotone equation (1) with the initial condition $\varphi(0, u, f) = u$ and let it be defined on $\mathbb{R}_+$. In virtue of the fundamental theory of ODEs with the holomorphic right hand side (see, for example, [9] and [11]) the mapping φ possesses the following properties:

1. $\varphi(0, u, f) = u$;
2. $\varphi(t + \tau, u, f) = \varphi(t, \varphi(\tau, f, z), f_\tau)$ for every $t, \tau \in \mathbb{R}^+$ and $u \in E$, where f_τ is a τ-translation of the function f;
3. φ is continuous;
4. $V(\varphi(t, u_1, f), \varphi(t, u_2, f)) \leq V(u_1, u_2)$ for every $t \in \mathbb{R}_+$ and $u_1, u_2 \in E$.

Properties 1.-4. will make the basis of our research of the abstract V-monotone non-autonomous dynamical system.

This paper is organized as follows.

Section 2 contains the notions of cocycles, skew-product dynamical systems and non-autonomous dynamical systems. We establish some properties of the set-valued mappings.

In section 3 we introduce the notion of V-monotone cocycles and establish one important property of this class of cocycles (Lemma 3.4).

Section 4 is devoted to the study of the continuous invariant sections of non-autonomous dynamical systems. Continuous invariant sections play a very important role in the study of the problem of the existence of almost periodic solutions of differential equations. This section contains the main results of the paper (Theorems 4.3 and 4.10), where we give the conditions of the existence of continuous invariant sections of V-monotone cocycles.

In section 5 we study the problem of the existence of periodic (quasi-periodic, almost periodic, almost automorphic, recurrent) motions of V-monotone cocycles with the compact minimal base that contains only periodic (quasi-periodic, almost periodic, almost automorphic, recurrent) motions. The main results of this section are Theorems 5.11 and 5.11.

Section 6 is devoted to the application of our general results obtained in sections 2-5 to the study of differential equations (ODEs, Caratheodory's equations with almost periodic coefficients, almost periodic ODEs with impulse and almost periodic difference equations).

2. Cocycles, skew-product dynamical systems and non-autonomous dynamical systems

Definition 2.1. *Let* $(X, \mathbb{T}_1, \pi)$ *and* $(Y, \mathbb{T}_2, \sigma)$ $(\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S})$ *be two dynamical systems. A mapping* $h : X \to Y$ *is called a homomorphism (isomorphism, respectively) of the dynamical system* $(X, \mathbb{T}_1, \pi)$ *on* $(Y, \mathbb{T}_2, \sigma)$*, if the mapping* h *is continuous (homeomorphic, respectively) and* $h(\pi(x, t)) = \sigma(h(x), t)$ $(\ t \in \mathbb{T}_1,\ x \in X)$*. In this case the dynamical system* $(X, \mathbb{T}_1, \pi)$ *is an extension of the dynamical system* $(Y, \mathbb{T}_2, \sigma)$ *by the homomorphism* h*, but the dynamical system* $(Y, \mathbb{T}_2, \sigma)$ *is called a factor of the dynamical system* $(X, \mathbb{T}_1, \pi)$ *by the homomorphism* h*. The dynamical system* $(Y, \mathbb{T}_2, \sigma)$ *is called also a base of the extension* $(X, \mathbb{T}_1, \pi)$*.*

Definition 2.2. *A triplet* $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma),\ h \rangle$*, where* h *is a homomorphism from* $(X, \mathbb{T}_1, \pi)$ *on* $(Y, \mathbb{T}_2, \sigma)$ *and* (X, h, Y) *is a locally trivial fibering, is called a non-autonomous dynamical system.*

Remark 2.3. *In the latter years in the works of I.U.Bronsteyn and his collaborators (see, for example,* [3]*) an extension is called a triplet* $\langle (X, \mathbb{T}, \pi),\ (Y, \mathbb{T}, h),\ h \rangle$*, i.e. the object which we call here a non-autonomous dynamical system.*

Definition 2.4. *A triplet* $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ *(or shortly* φ*), where* $(Y, \mathbb{T}_2, \sigma)$ *is a dynamical system on* Y*,* W *is a complete metric space and* φ *is a continuous mapping from* $\mathbb{T}_1 \times W \times Y$ *in* W*, possessing the following conditions:*

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$*;*
b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1,\ u \in W, y \in Y)$,

is called [20] *a cocycle on* $(Y, \mathbb{T}_2, \sigma)$ *with the fiber* W*.*

Definition 2.5. *Let* $X := W \times Y$ *and define a mapping* $\pi : X \times \mathbb{T}_1 \to X$ *as following:* $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$ *(i.e.* $\pi = (\varphi, \sigma)$*). Then it is easy to see that* $(X, \mathbb{T}_1, \pi)$ *is a dynamical system on* X *which is called a skew-product dynamical system* [20] *and* $h = pr_2 : X \to Y$ *is a homomorphism from* $(X, \mathbb{T}_1, \pi)$ *on* $(Y, \mathbb{T}_2, \sigma)$ *and, consequently,* $\langle (X, \mathbb{T}_1, \pi),\ (Y, \mathbb{T}_2, \sigma), h \rangle$ *is a non-autonomous dynamical system.*

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on the dynamical system $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi),$ $(Y, \mathbb{T}_2, \sigma), h \rangle$ $(X := W \times Y)$ called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system). Below we give an example of this.

Example 2.6. Let E be a n-dimensional real or complex Euclidean space. Let us consider a differential equation

$$u' = f(t,u), \tag{2}$$

where $f \in C(\mathbb{R} \times E, E)$. Along with equation (2) we consider its H-class [3],[14], [20], [23], i.e. the family of equations

$$v' = g(t,v), \tag{3}$$

where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}$, $f_\tau(t,u) = f(t+\tau,u)$ for all $(t,u) \in \mathbb{R} \times E$ and by bar we denote the closure in $C(\mathbb{R} \times E, E)$. We will suppose also that the function f is regular, i.e. for every equation (3) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(\cdot, v, g)$ the solution of equation (3) passing through the point $v \in E$ at the initial moment $t = 0$. Then there is a correctly defined mapping $\varphi : \mathbb{R}_+ \times E^n \times H(f) \to E$ satisfying the following conditions (see, for example, [3], [20]):

1) $\varphi(0,v,g) = v$ for all $v \in E$ and $g \in H(f)$;
2) $\varphi(t,\varphi(\tau,v,g),g_\tau) = \varphi(t+\tau,v,g)$ for every $v \in E$, $g \in H(f)$ and $t,\tau \in \mathbb{R}_+$;
3) the mapping $\varphi : \mathbb{R}_+ \times E \times H(f) \to E$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}_+, \sigma)$ a dynamical system of translations (a semi-group system) on Y, induced by the dynamical system of translations $(C(\mathbb{R} \times E^n, E^n), \mathbb{R}, \sigma)$. The triplet $\langle E, \varphi, (Y, \mathbb{R}_+, \sigma)\rangle$ is a cocycle on $(Y, \mathbb{R}_+, \sigma)$ with the fiber E. Thus, equation (2) generates a cocycle $\langle E, \varphi, (Y, \mathbb{R}_+, \sigma)\rangle$ and a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$, where $X := E^n \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$.

Definition 2.7. *A family of subsets $\{I_y \mid y \in Y\}$ ($I_y \subseteq E$) is called positively invariant (negatively invariant, invariant) w.r.t. a cocycle φ if $\varphi(t, I_y, y) \subseteq I_{yt}$ ($\varphi(t, I_y, y) \supseteq I_{yt}$, $\varphi(t, I_y, y) =\subseteq I_{yt}$) for all $t \in \mathbb{T}_+$ and $y \in Y$, where $yt := \sigma(t,y)$.*

Lemma 2.8. *The family of subsets $\{I_y \mid y \in Y\}$ is positively invariant (negatively invariant, invariant) w.r.t. the cocycle φ if and only if the set $J := \bigcup\{J_y \mid y \in Y\}$, where $J_y := I_y \times y$, is positively invariant (negatively invariant, invariant) w.r.t. the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ ($X := E \times Y$ and $\pi := (\varphi, sigma)$).*

Proof. This statement follows directly from the corresponding definitions and the equality

$$\pi^t J = \bigcup\{\pi^t J_y \mid y \in Y\} = \bigcup\{(\varphi(t, I_y, y), yt) \mid y \in Y\}.$$

□

Let $M \subseteq E$ be an arbitrary subset of E. Denote by $M_y := \{x \in M \mid$. There exists $y \in Y$ such that $\varphi(t,x,y) \in M$ for all $t \in \mathbb{S}_+\}$ and $M_{inv} := \bigcup\{M_y \mid y \in Y\}$.

Remark 2.9. *The family of subsets $\{M_y \mid y \in Y\}$ is positively invariant w.r.t. the cocycle φ.*

Lemma 2.10. *For the arbitrary subset $M \subseteq E$ the following statements are equivalent:*

(i) $M = M_{inv}$;

(ii) *the family of subsets* $\{M_y \mid y \in Y\}$ *is positively invariant.*

Proof. This affirmation is straightforward. □

Lemma 2.11. *Let* Y *be compact and* $M \subseteq E$ *be closed (i.e.* $\overline{M} = M$, *where by bar we denote the closure in the space* E*). Then the set* M_{inv} *is closed too.*

Proof. Let $x \in \overline{M}_{inv}$. Then there exists a sequence $\{x_n\} \subseteq M_{inv}$ and $y_n \in Y$ such that $x_n \in M_{y_n}$ and $x = \lim_{n\to+\infty} x_n$. Taking into account that the set Y is compact ,we can suppose that the sequence $\{y_n\}$ is convergent and denote by $y = \lim_{n\to+\infty} y_n$. Since $x_n \in M_{y_n}$, we have

$$\varphi(t, x_n, y_n) \in M \tag{4}$$

for all $t \in \mathbb{S}_+$. Passing to the limit in inclusion (4) as $n \to +\infty$, we obtain $\varphi(t, x, y) \in M$ for all $t \in \mathbb{S}_+$, i.e. $x \in M_y \subseteq M_{inv}$. The lemma is proved. □

Let $A, B \subseteq E$ be two bounded subsets and $\beta(A, B) := \sup_{a\in A} \rho(a, B)$ be a semi-distance of Hausdorff.

Definition 2.12. *A set-valued mapping* $F : Y \to B(E)$ *(*$B(E)$ *is the family of all bounded subsets of* E*) is called upper semi-continuous (lower semi-continuous) in* $y_0 \in Y$*, if* $\lim_{y\to y_0} \beta(F(y), F(y_0)) = 0$ *(*$\lim_{y\to y_0} \beta(F(y_0), F(y)) = 0$ *). If the set-valued mapping* $F : Y \to B(E)$ *is upper and lower semi-continuous, then it is called continuous.*

Lemma 2.13. *Let* $\{M_y \mid y \in Y\}$ *be the family of subsets of* E *possessing the following properties:*

(i) *the set* $M := \bigcup\{M_y \mid y \in Y\}$ *is relatively compact;*
(ii) *the set-valued mapping* $y \to M_y$ *is lower semi-continuous;*
(iii) $M_y \neq \emptyset$ *and is compact for all* $y \in Y$.

Then the set-valued mapping $y \to K_y := \bigcap\{B_V(p, d) \mid p \in M_y\}$, *where* $d := diamM$ *and* $B_V(p, d) := \{x \in E \mid V(x, p) \leq d\}$, *is upper semi-continuous.*

Proof. Let $y \in Y,\ y_n \to y,\ x_n \in K_{y_n}$ and $x_n \to x$. We will show that $x \in K_y$, i.e. $\rho(x, p) \leq d$ for all $p \in M_y$. In fact, since $x_n \in K_{y_n}$, we have $\rho(x_n, p) \leq d$ for all $p \in M_{y_n}$.

Let $p \in K_y$ be an arbitrary point. Taking into consideration the lower semi-continuity of the set-valued mapping $y \to K_y$, we have two sequences $\{y_n\} \subseteq Y$ and $\{p_n\} \subseteq E$ such that $p_n \in K_{y_n}$, $p_n \to p$ and $y_n \to y$. Thus,

$$\rho(x_n, p_n) \leq d \tag{5}$$

for all $n \in \mathbb{N}$. Passing to the limit in inequality (5), we obtain $\rho(x, p) \leq d$ for all $p \in M_y$. This means that $x \in K_y$. The lemma is proved. □

3. $V-$monotone cocycles

Definition 3.1. *A cocycle φ is called $\mathcal{V}-$monotone (see* [5], [14], [24]*), if there exists a continuous function $\mathcal{V} : E \times E \to \mathbb{R}_+$ with the following properties:*

a. $\mathcal{V}(u_1, u_2) \geq 0$ *for all* $u_1, u_2 \in E$*;*
b. $\mathcal{V}(u_1, u_2) = 0$ *if and only if* $u_1 = u_2$*;*
c. *the function V is symmetric, i.e.* $V(x_1, x_2) = V(x_2, x_1)$ *for all* $x_1, x_2 \in E$*;*
d. *the function V is convex, i.e.* $V(\lambda x_1 + (1-\lambda)x_2, p) \leq \lambda V(x_1, p) + (1-\lambda)V(x_2, p)$ *for all* $x_1, x_2, p \in E$ *and* $\lambda \in [0, 1]$*;*
e. $V(x_1, x_2) \leq V(x_1, x_3) + V(x_3, x_2)$ *for all* $x-1, x_2, x_3 \in E$*;*
f. *there are two continuous, positively defined and strictly increasing functions* $a, b : \mathbb{S}_+ \to \mathbb{R}_+$ *such that* $a(0) = b(0) = 0$, $Im(a) = Im(b)$ $(Im(a) := a(\mathbb{S}_+))$ *and* $a(\rho(u_1, u_2) \leq \mathcal{V}(u_1, u_2) \leq b(\rho(u_1, u_2))$ *for all* $u_1, u_2 \in E$, *where* $\rho(u_1, u_2) := |u_1 - u_2|$ *is a distance on* E*;*
g. $\mathcal{V}(\varphi(t, u_1, y), \varphi(t, u_2, y)) \leq \mathcal{V}(u_1, u_2)$ *for all* $u_1, u_2 \in E$, $y \in Y$ *and* $t \in \mathbb{S}_+$.

Remark 3.2. *1. From the conditions c.-e. follows that by the equality* $d(u_1, u_2) := V(u_1, u_2)$ *is defined some distance on the space* E.

2. According to condition f. the distance ρ and d on the space E are topologically equivalent.

Lemma 3.3. [5] *The cocycle φ is $\mathcal{V}-$monotone if and only if the non-autonomous dynamical system* $\langle (X, \mathbb{S}_+, \pi), (\Omega, \mathbb{S}, \sigma), h\rangle$ *generated by the cocycle φ is $V-$monotone, where* $V((u_1, y), (u_2, y)) := \mathcal{V}(u_1, u_2)$ *for all* $(u_i, y) \in X$ $(i = 1, 2)$.

Lemma 3.4. *Let φ be a V–monotone cocycle and* $\{M_y \mid y \in Y\}$ *be a family of subsets from E and the following conditions be held:*

(i) $\{M_y \mid y \in Y\}$ *is negatively invariant w.r.t. the cocycle* φ*;*
(ii) *the set* $M := \bigcup\{M_y \mid y \in Y\}$ *is compact, i.e. it is bounded and closed.*

Then there exists a relatively compact subset K of E possessing the following properties:

(i) $M_y \subseteq K_y$ *for all* $y \in Y$*;*
(ii) *the set K_y is convex for any* $y \in Y$*;*
(iii) *the family of subsets* $\{K_y \mid y \in Y\}$ *is positively invariant w.r.t. the cocycle* φ.

Proof. Let $y \in Y$ and

$$K_y := \bigcap\{B_V(p, d) \mid p \in M_y\},$$

where $d := \sup\{V(x_1, x_2) \mid x_1, x_2 \in M\}$ and $B_V(p, d) := \{q \in E \,| V(q, p) \leq d\}$. It is clear that the set K_y is bounded, closed, convex and $M_y \subseteq K_y$. We will prove that the family $\{K_y \mid y \in Y\}$ is positively invariant w.r.t. the cocycle φ. In fact, if $y \in Y$ and $x \in K_y$, then $V(x, p) \leq d$ for all $p \in M_y$. Since the cocycle φ is V–monotone, we have

$$V(\varphi(t, x, y), \varphi(t, p, y)) \leq V(x, p) \leq d$$

for all $p \in B_V(x,d)$, i.e. $\varphi(t,x,y) \in B_V(\varphi(t,p,y),d)$ for all $x \in B_V(p,d)$ and $p \in M_y$ and, consequently, $\varphi(t,B_V(p,d),y) \subseteq B_V(\varphi(t,p,y),d)$. According to the conditions of the lemma, $\varphi(t,M_y,y) \supseteq M_{yt}$ and hence we have

$$\begin{aligned} U(t,y)K_y &= U(t,y)(\bigcap\{B_V(p,d) \mid p \in M_y\}) \\ &\subseteq \bigcap\{U(t,y)B_V(p,d) \mid p \in M_y\} \\ &\subseteq \bigcap\{B_V(U(t,y)p,d) \mid p \in M_y\} \\ &\subseteq \bigcap\{B_V(q,d) \mid q \in M_{yt}\} = K_{yt} \end{aligned}$$

for all $y \in Y$ and $t \in \mathbb{S}_+$.

Now we note that $K \subseteq B(M,d) := \{x \in E \mid \sup\limits_{p \in M} \rho(p,x) \leq d\}$ and, consequently, it is bounded. □

4. Invariant sections of non-autonomous dynamical systems

Let $\langle (X,\mathbb{S}_+,\pi),\ (Y,\mathbb{S},\sigma),h)\rangle$ be a non-autonomous dynamical system.

Definition 4.1. *A mapping $\gamma : Y \to X$ is called a section (selector) of a homomorphism h, if $h(\gamma(y)) = y$ for all $y \in Y$. The section γ of the homomorphism h is called invariant, if $\gamma(\sigma(t,y)) = \pi(t,\gamma(y))$ for all $y \in Y$ and $t \in \mathbb{S}$.*

Denote by $\Gamma = \Gamma(Y,X)$ the family of all continuous sections of h, i.e. $\Gamma(Y,X) = \{\gamma \in C(Y,X) :\ h \circ \gamma = Id_Y\}$. We will suppose that $\Gamma(Y,X) \neq \emptyset$. This condition is fulfilled in many important cases for the applications.

Remark 4.2. *A continuous section $\gamma \in \Gamma$ is invariant if and only if $\gamma \in \Gamma$ is a stationary point of the semigroup $\{S^t \mid t \in \mathbb{S}_+\}$, where $S^t : \Gamma(Y,X) \to \Gamma(Y,X)$ is defined by the equality $(S^t\gamma)(y) := \pi(t,\gamma(\sigma(-t,y)))$ for all $y \in Y$ and $t \in \mathbb{S}_+$.*

We consider a special case of the foregoing construction. Let $\langle W,\varphi,\ (Y,\mathbb{S},\sigma)\rangle$ be a cocycle over $(Y,\mathbb{S},\sigma)$ with the fiber W and $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ be the non-autonomous dynamical system generated by this cocycle. Then $h \circ \gamma = Id_Y$ and since $h = pr_2$, then $\gamma = (\psi, Id_Y)$, where $\gamma \in \Gamma(Y,X)$ and $\psi : Y \to W$. Hence, to each section γ there corresponds a mapping $\psi : Y \to W$ and conversely. There being a one-to-one relation between $\Gamma(Y, W \times Y)$ and $C(Y,W)$, where $C(Y,W)$ is the space of continuous functions $\psi : Y \to W$, we identify these two objects from now on. The semigroup $\{S^t \mid t \in \mathbb{S}_+\}$ naturally induces a semigroup $\{Q^t \mid t \in \mathbb{S}_+\}$ of the mappings of $C(Y,W)$. Namely,

$$\begin{aligned} &(S^t\gamma)(y) = \pi^t\gamma(\sigma^{-t}y) = \pi^t(\psi, Id_Y)(\sigma^{-t}y) = \\ &\pi^t(\psi(\sigma^{-t}y),\sigma^{-t}y) = (U(t,\sigma^{-t}y)\psi(\sigma^{-t}y),y) = ((Q^t\psi)(y),y), \end{aligned}$$

where $U(t,y) := \varphi(t,\cdot,y)$.

Hence, $S^t(\psi,Id_Y) = (Q^t\psi, Id_Y)$ with $(Q^t\psi)(y) = U(t,\sigma^{-t}y)\psi(\sigma^{-t}y)\ \ (y \in Y)$. We have the following properties:

a. $Q^0 = Id_{C(Y,W)}$;
b. $Q^tQ^\tau = Q^{t+\tau}\ (\ t,\tau \in \mathbb{S}_+)$.

Theorem 4.3. *Let $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a V–monotone cocycle on $(Y, \mathbb{S}, \sigma)$ with the fiber E, $K \subseteq E$ be a bounded subset of E and the following conditions be held:*

(i) *Y is compact;*
(ii) *the family of subsets $\{K_y \mid y \in Y\}$ is positively invariant w.r.t. the cocycle φ;*
(iii) *the set K_y is nonempty, compact and convex for every $y \in Y$;*
(iv) *the set-valued map $y \to K_y$ is upper continuous.*

Then there exists at least one continuous invariant section of the cocycle φ.

Proof. Denote by $\mathcal{K} := \{\psi \mid \psi \in C(Y, E)$ such that $\psi(y) \in K_y\}$, where $C(Y, E)$ is a space of all continuous functions $f : Y \to E$ equipped with the sup–norm. Note that the set $\mathcal{K}$ possesses the following properties:

(i) the set $\mathcal{K}$ is not empty according to Theorem of Michael (see, for example, [1, 2]);
(ii) $\mathcal{K}$ is closed, bounded and convex in the space $C(Y, E)$;
(iii) $\mathcal{K}$ is invariant with respect to the semigroup $\{Q^t \mid t \in, \mathbb{S}_+\}$ of the mappings of $C(Y, E)$, i.e. $Q^t\mathcal{K} \subseteq \mathcal{K}$ for all $t \in \mathbb{S}_+$;
(iv) $\mathcal{K}$ is relatively compact with respect to the weak topology of $C(Y, E)$ (see, for example, [13, Chapter7]).

According to Theorem 3.12 [17] the weak closure $\overline{\mathcal{K}}_w$ of the set $\mathcal{K}$ coincides with its closure in the topology of the space $C(Y, E)$. Since $\mathcal{K}$ is closed in $C(Y, E)$, we have $\overline{\mathcal{K}}_w = \mathcal{K}$, i.e. the set $\mathcal{K}$ is weakly compact.

Now we will prove that the mapping $Q : \mathbb{S}_+ \times \mathcal{K} \to \mathcal{K}$ $(Q(t, \gamma) = Q^t(\gamma))$ for all $t \in \mathbb{S}_+$ and $\gamma \in \mathcal{K}$) is weakly continuous. Really, at first we note that the mapping $Q^t : \mathcal{K} \to \mathcal{K}$ $(t \in \mathbb{S}_+)$ is weakly continuous. Let $\gamma_n \rightharpoonup \gamma$ (the sequence $\{\gamma_n\}$ weakly converges to γ), then the sequence $\{\gamma_n\}$ is uniformly bounded and $\gamma_n(y)$ converges to $\gamma(y)$ in E for every $y \in Y$ (see, for example, []). Thus we have that $\gamma_n(\sigma(-t, y))$ converges to $\gamma(\sigma(-t, y))$ in E (for any $y \in Y$) and according to the continuity of the mapping $U(t, \sigma(-t, y)) : E \to E$ we obtain $U(t, \sigma(-t, y))\gamma_n(\sigma(-t, y) \to U(t, \sigma(-t, y)\gamma(\sigma(-t, y)$, i.e. $Q^t\gamma_n(y) \to Q^t\gamma(y)$ (for every $y \in Y$). On the other hand, $\{Q^t\gamma_n\} \subseteq \mathcal{K}$ and, consequently, the sequence $\{Q^t\gamma_n\}$ is uniformly bounded. Thus we have $Q^t\gamma_n \rightharpoonup Q^t\gamma$.

Let now $t_n \to 0$ and $\gamma_n \rightharpoonup \gamma$. Then

$$\begin{aligned}
&V(Q^{t_n}\gamma(y), \gamma(y)) = V(U(t_n, \sigma(-t_n, y))\gamma(\sigma(-t_n, y), \gamma(y)) \leq \\
&V(U(t_n, \sigma(-t_n, y))\gamma(\sigma(-t_n, y), U(t_n, \sigma(-t_n, y))\gamma(y)) \leq \\
&V(U(t_n, \sigma(-t_n, y))\gamma(y), \gamma(y)) \leq V(\gamma(\sigma(-t_n, y), \gamma(y)) + \\
&V(U(t_n, \sigma(-t_n, y))\gamma(y), \gamma(y)) \to 0
\end{aligned}$$

as $n \to +\infty$, because the functions φ and $\gamma \in \overline{\mathcal{K}}_w = \mathcal{K}$ are continuous. Since under the conditions of Theorem the distances $d(x_1, x_2) := V(x_1, x_2)$ and $\rho(x_1, x_2) := |x_1 - x_2|$ are topologically equivalent, then $Q^{t_n}\gamma(y) \to \gamma(y)$ in E (for every $y \in Y$).

If $t_n \to t_0$ and $\gamma_n \rightharpoonup \gamma$ (i.e. γ is a weak limit $(w - \lim)$ of the sequence $\{\gamma_n\}$), then

$$|Q^{t_n}\gamma_n(y) - Q^{t_0}\gamma(y)| \leq |Q^{t_n}\gamma_n(y) - Q^{t_0}\gamma_n(y)| + |Q^{t_0}\gamma_n(y) - Q^{t_0}\gamma(y)| \to 0$$

as $n \to +\infty$, according to the facts established above.

Thus, the triplet $(\mathcal{K}, \mathbb{S}_+, Q)$ (the space $\mathcal{K}$ is equipped by the weak topology of $C(Y,E)$) is a semigroup dynamical system and $\mathcal{K}$ is a weakly compact convex subset of the Banach space $C(Y,E)$.

Let now $t_k \downarrow 0$. Then according to Theorem of Schauder-Tihonoff (see, for instance, [12]) the mapping $Q^{t_n} : \mathcal{K} \to \mathcal{K}$ admits at least one fixed point $\gamma_k \in \mathcal{K}$. Since $\mathcal{K}$ is weakly compact, then without loss of generality we can consider that the sequence $\{\gamma_k\}$ is weakly convergent. Denote by $\gamma := w - \lim\limits_{k\to+\infty} \gamma_k$ and let $t \in \mathbb{S}_+$ be an arbitrary number, then for every t_k there exists $n_k \in \mathbb{Z}_+$ and $\tau_k \in [0, t_k)$ such that $t = t_k n_k + \tau_k$ and, consequently, $Q^t\gamma = w - \lim\limits_{k\to+\infty} Q^{t_k n_k+\tau_k}\gamma_k = w - \lim\limits_{k\to+\infty} Q^{\tau_k}\gamma_k = \gamma$. The theorem is proved. □

Remark 4.4. *We presented the proof of Theorem 4.3 for the case when $\mathbb{S}_+ = \mathbb{R}_+$. The case when $\mathbb{S}_+ = \mathbb{Z}_+$ is analogous and simpler.*

A function $\gamma_{(u,y)} : \mathbb{R} \to W$ represents the entire trajectory $\gamma_{(u,y)}$ of a cocycle $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ if $\gamma_{(u,y)}(0) = u \in W$ and $\varphi(t, \gamma_{(u,y)}(\tau), \sigma_\tau\omega) = \gamma_{(u,y)}(t+\tau)$ for $t \in \mathbb{S}_+$ and $\tau \in \mathbb{S}$.

Definition 4.5. *Let $\{K_y \mid y \in Y\}$ be an invariant set of the cocycle φ. The cocycle φ is called distal on $\{K_y \mid y \in Y\}$ in the negative direction, if*

$$\inf_{t\in\mathbb{S}_-} \rho\left(\gamma_{(u_1,y)}(t), \gamma_{(u_2,y)}(t)\right) > 0$$

for any entire trajectories $\gamma_{(u_1,y)}$ and $\gamma_{(u_2,y)}$ with $(u_1, y), (u_2, y) \in K_y$ $(u_1 \neq u_2)$ and any $y \in Y$.

Recall that an autonomous dynamical system $(Y, \mathbb{S}, \sigma)$ is called minimal if Y does not contain proper compact subsets which are σ-invariant.

The following lemma is due to Furstenberg (see, for example, [3, Chapter 3] or [14, Chapter 7] Proposition 4).

Lemma 4.6. *Suppose that the cocycle $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is distal on $\mathbb{S}_-$ and that $(Y, \mathbb{S}, \sigma)$ is compact and minimal. In addition, suppose that a compact subset J of X is π–invariant with respect to the skew–product system $(X, \mathbb{S}_+, \pi)$. Then the set-valued mapping $y \to I_y := \{u \in E : (u, y) \in J\}$ is continuous.*

Definition 4.7. *Let $\{K_y \mid y \in Y\}$ be a positive invariant set of the cocycle φ. A cocycle φ is said to be uniformly Lyapunov stable on $\{K_y \mid y \in Y\}$, if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$\rho\left(\varphi(t, u_1, y), \varphi(t, u_2, y)\right) < \varepsilon$$

for all u_1, $u_2 \in K_y$ with $\rho(u_1, u_2) < \delta$, $y \in Y$ and $t \geq 0$.

Lemma 4.8. *Let $\{K_y \mid y \in Y\}$ be an invariant set of the V-monotone cocycle φ. Then φ is distal on $\{K_y \mid y \in Y\}$ in the negative direction.*

Proof. We note that from the V-monotonicity of the cocycle φ it follows its uniform Lyapunov stability in the positive direction. Really, let $\varepsilon > 0$ and $\delta(\varepsilon) := a^{-1}(b(\varepsilon))$

(a and b are the functions from the definition of the V-monotonicity of the cocycle φ). Then it is easy to see that the inequality $\rho(u_1,u_2)<\delta$ ($u_1,u_2\in K_y$) implies $\rho(\varphi(t,u_1,y),\varphi(t,u_2,y))<\varepsilon$ for all $t\in\mathbb{S}_+$ and $y\in Y$.

Now we will prove that the cocycle φ is distal. Suppose that it is not distal. Then there is $y_0\in Y$, a sequence $t_n\to\infty$ and entire trajectories $\gamma_{(u_1,y_0)}$, $\gamma_{(u_2,y_0)}$ with $u_1 \neq u_2$ such that

$$\lim_{n\to\infty}\rho\left(\gamma_{(u_1,y_0)}(-t_n),\gamma_{(u_2,y_0)}(-t_n)\right)=0.$$

Let $\varepsilon=\rho\left(u_1,u_2\right)>0$ and choose $\delta=\delta(\varepsilon)>0$ from the property of the uniform Lyapunov stability. Then

$$\rho\left(\gamma_{(u_1,\omega_0)}(-t_n),\gamma_{(u_2,\omega_0)}(-t_n)\right)<\delta$$

for sufficiently large n. Hence

$$\rho\left(\varphi(t,\gamma_{(u_1,\omega_0)}(-t_n),\sigma_{-t_n}\omega_0),\varphi(t,\gamma_{(u_2,\omega_0)}(-t_n),\sigma_{-t_n}\omega_0)\right)<\varepsilon$$

for $t\geq 0$ and, in particular,

$$\varepsilon=\rho(u_1,u_2)=\rho(\varphi(t_n,\gamma_{(u_1,\omega_0)}(-t_n),\sigma_{-t_n}\omega_0),$$
$$\varphi(t_n,\gamma_{(u_2,\omega_0)}(-t_n),\sigma_{-t_n}\omega_0))<\varepsilon$$

for $t=t_n$. And this is a contradiction. The lemma is proved. □

Lemma 4.9. *Let $\langle E,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be a V-monotone cocycle, $(Y,\mathbb{S},\sigma)$ be compactly minimal and $\{K_y \mid y\in Y\}$ be invariant w.r.t. the cocycle φ and $\bigcup\{K_y\mid y\in Y\}$ be relatively compact. Then the set-valued mapping $y\to K_y$ is continuous.*

Proof. This statement directly follows from Lemmas 4.6 and 4.8. □

Theorem 4.10. *Let $\langle E,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be a V-monotone cocycle, Y be a compact minimal set there are points $y_0\in Y$ and $x_0\in E$ such that $\varphi(t,x_0,y_0)$ is bounded on $\mathbb{S}_+$.*

Then the cocycle φ admits at least one continuous invariant section.

Proof. Consider the non-autonomous dynamical system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$, where $(X,\mathbb{S}_+,\pi)$ is a skew-product dynamical system (i.e. $X:=E\times Y$ and $\pi:=(\varphi,\sigma)$ and $h:=pr_2:Y\to X$). Since the space E is finite-dimensional, the trajectory of the dynamical system $(X,\mathbb{S}_+,\pi)$ passing through the point $x_0:=(u_0,y_0)$ is relatively compact on $\mathbb{S}_+$ and, consequently, the ω-limit set ω_{x_0} of x_0 is nonempty, compact and invariant. According to Theorem of Birkhoff the set ω_{x_0} contains at least one minimal set $\mathcal{M}\subseteq\omega_{x_0}$. Since Y is compact and minimal, then we have $h(\mathcal{M})=Y$. Let $M:=pr_1(\mathcal{M})$, where pr_1 is the first projection of X to E, and $M_y:=pr_1(\mathcal{M}_y)$ ($\mathcal{M}_y:=h^{-1}(y)\bigcap X$). Then the family of subsets $\{M_y\mid y\in Y\}$ is invariant w.r.t. the cocycle φ, since the set $\mathcal{M}$ is invariant w.r.t. the dynamical system $(X,\mathbb{S}_+,\pi)$. By Lemma 4.9 the set-valued mapping $y\to M_y$ is continuous.

If the set M_{y_0} consists of a single point, then every set M_y contains exactly one point u_y. It is easy to see that the function $\gamma: Y\to E$ defined by the equality $\gamma(y):=u_y$ is the desired continuous invariant section of the cocycle φ.

Let now the set M_{y_0} contain more than one point. Then by the continuity of the mapping $y\to M_y$ under the conditions of the theorem every subset M_y contains more than one point and, consequently, $d \geq d_y := diamM_y > 0$ ($d := diamM$) for all $y \in Y$. By Lemma 2.13 the set-valued mapping $y \to K_y := \bigcap\{B_V(p,d) \mid p \in M_y\}$, where $B_V(p,d) := \{u \in E \mid V(u,p) \leq d\}$, is upper semi-continuous and possesses the following properties:

(i) $K_y \neq \emptyset$, is compact and convex;
(ii) the family of subsets $\{K_y \mid y \in Y\}$ is positively invariant w.r.t. the cocycle φ.

According to Theorem 4.3 the cocycle φ admits at least one continuous invariant section. The theorem is proved. □

Definition 4.11. *The cocycle φ is called dissipative if there exists a positive number r such that $\limsup_{t\to+\infty} |\varphi(t,u,y)| \leq r$ for all $u \in E$ and $y \in Y$.*

Lemma 4.12. *Let φ be a dissipative V-monotone cocycle. Then it admits at least one continuous invariant section.*

Proof. This affirmation follows from Theorem 4.10 because every motion $\varphi(t,u,y)$ of the cocycle φ is bounded on $\mathbb{S}_+$. □

5. Recurrent, almost periodic and almost automorphic motions

In this section, we discuss the problem of the existence of recurrent, almost periodic and almost automorphic motions of V-monotone cocycles.

Let $(X,\mathbb{T},\pi)$ be a dynamical system.

Definition 5.1. *A point $x \in X$ is called a τ-periodic, $\tau > 0, \tau \in \mathbb{T}$) point, if $xt = x$ ($x\tau = x$ respectively) for all $t \in \mathbb{T}$, where $xt := \pi(t,x)$.*

Definition 5.2. *A number $\tau \in \mathbb{T}$ is called an $\varepsilon > 0$ shift (almost period), if $\rho(x\tau,x) < \varepsilon$ (respectively $\rho(x(\tau+t),xt) < \varepsilon$ for all $t \in \mathbb{T}$).*

Definition 5.3. *A point $x \in X$ is called almost recurrent (almost periodic), if for any $\varepsilon > 0$ there exists a positive number l such that on any segment of length l there is an ε shift (almost period) of point $x \in X$.*

Definition 5.4. *If the point $x \in X$ is almost recurrent and the set $H(x) = \overline{\{xt \mid t \in \mathbb{T}\}}$ is compact, then x is called recurrent.*

Let $\mathbb{T} = \mathbb{S}$ and $(X,\mathbb{S},\pi)$ be a bi-sided dynamical system.

Definition 5.5. *A recurrent point $x \in X$ is called almost automorphic if whenever t_α is a net with $xt_\alpha \to x_*$, then $x_*(-t_\alpha) \to x$ too.*

Definition 5.6. *A motion $\varphi(t,u_0,y_0)$ ($u_0 \in E$ and $y_0 \in Y$) of the cocycle φ is called recurrent (almost periodic, almost automorphic, quasi-periodic, periodic), if the point $x_0 := (u_0,y_0) \in X := E \times Y$ is a recurrent (almost periodic, almost automorphic, quasi-periodic, periodic) point of the skew-product dynamical system $(X,\mathbb{S}_+,\pi)$ ($\pi := (\varphi,\sigma)$).*

Remark 5.7. *We note (see, for example,* [14] *and* [23]*) that if $y \in Y$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ onto $(X, \mathbb{T}_1, \pi)$, then the point $x = h(y)$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the system $(X, \mathbb{T}_1, \pi)$.*

Lemma 5.8. *If $y \in Y$ is an almost automorphic point of the dynamical system $(Y, \mathbb{S}, \sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{S}, \sigma)$ onto $(X, \mathbb{S}_+, \pi)$, then the point $x = h(y)$ is an almost automorphic point of the system $(X, \mathbb{S}_+, \pi)$.*

Proof. Let t_α be a net with $xt_\alpha \to x_*$, then we have $yt_\alpha \to y_*$ ($y := h(x)$ and $y_* := h(x_*)$). Since the point y is almost automorphic, then also $y_*(-t_\alpha) \to y$ and, consequently, $x_*(-t_\alpha) = h(y_*(-t_\alpha)) \to h(y) = x$. The lemma is proved. □

Remark 5.9. *Let $X := E \times Y$ and $\pi := (\varphi, \sigma)$. Then mapping $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ onto $(X, \mathbb{T}_1, \pi)$ if and only if $h(y) = (\gamma(y), y)$ for all $y \in Y$, where $\gamma : Y \to E$ is a continuous mapping with the condition that $\gamma(yt) = \varphi(t, \gamma(y), y)$ for all $y \in Y$ and $t \in \mathbb{T}_2$.*

Theorem 5.10. *Let Y be a compact minimal set and $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a V-monotone cocycle. Under the conditions of Theorem 4.10 the cocyle φ admits at least one continuous invariant section $\gamma : Y \to E$ and the motion $\varphi(t, \gamma(y), y)$ will be stationary (τ-periodic, quasi-periodic, almost periodic, almost automorphic, recurrent) if the point $y \in Y$ is so.*

Proof. This statement directly follows from Theorem 4.10, remark 5.7 and Lemma 5.8. □

Theorem 5.11. *Let Y be a compact minimal set and $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a V-monotone dissipative cocycle. Then the cocyle φ admits at least one continuous invariant section $\gamma : Y \to E$ and the motion $\varphi(t, \gamma(y), y)$ will be stationary (τ-periodic, quasi-periodic, almost periodic, almost automorphic, recurrent) if the point $y \in Y$ is so.*

Proof. This statement directly follows from Theorem 5.10 and Lemma 4.12. □

6. Applications

6.1. ODEs.

Definition 6.1. *A function $W \in C(E, E)$ is called homogeneous (of order k, $k \geq 1$), if $W(\lambda x) = \lambda^k W(x)$ for all $x \in E$ and $\lambda \in \mathbb{R}_+ \setminus 0$.*

Denote by E a finite-dimensional Euclidean space with the scalar product $\langle, \rangle$ and the norm $|\cdot|$ generated by the scalar product. Let $[E]$ be the space of all the linear mappings $A : E \to E$ equipped with the operational norm.

Theorem 6.2. *Let Y be a compact minimal set, $F \in C(Y \times E, E)$ and $V \in C(E \times E, \mathbb{R}_+)$ and the following conditions be held:*

(i) *$V(x_1, x_2) := W(x_1 - x_2)$, where $W \in C(E, \mathbb{R}_+)$ is homogeneous and convex on E;*
(ii) *$W(x) = 0$ if and only if $x = 0$;*
(iii) *there exists $y_0 \in Y$ such that the equation*

$$u' = f(y_0 t, u) \quad (y_0 t =: \sigma(t, y_0)) \tag{6}$$

admits at least one bounded on $\mathbb{R}_+$ solution $\varphi(t, u_0, y_0)$;
(iv) *the cocycle φ generated by the family of equations*

$$u' = f(yt, u) \quad (y \in Y)) \tag{7}$$

is V-monotone.

Then there exists a continuous function $\gamma : Y \to E$ such that $\gamma(yt) = \varphi(t, \gamma(y), y)$ for all $y \in Y$ and $t \in \mathbb{R}$. If Y is a compact minimal set containing only τ-periodic (quasi periodic, almost periodic, almost automorphic, recurrent) motions, then equation (6) admits at least one τ-periodic (quasi periodic, almost periodic, almost automorphic, recurrent) solution $\varphi(t, \gamma(y), y)$.

Proof. Denote by $\alpha := \min_{|u|=1} W(u)$ and $\beta := \max_{|u|=1} W(u)$. Under the conditions of the theorem we have $\beta \geq \alpha > 0$ and, consequently, $\alpha|u_1 - u_2|^k \leq V(u_1, u_2) \leq \beta|u_1 - u_2|^k$ for all $u_1, u_2 \in E$. Now to finish the proof of the theorem it is sufficient to refer to Theorems 4.10 and 5.11. □

Example 6.3. As an example that illustrates this theorem we can consider the following equation

$$u' = g(u) + f(\sigma_t y),$$

where $f \in C(Y, \mathbb{R})$ and

$$g(u) = \begin{cases} (u+1)^2 & : \quad u < -1 \\ 0 & : \quad |u| \leq 1 \\ -(u-1)^2) & : \quad u > 1. \end{cases}$$

All solutions of this equation are bounded on $\mathbb{R}_+$ (see, for example, [**?**, Chapt.12]) and this equation is V-monotone, where $V(x_1, x_2) = |x_1 - x_2|^2$.

Example 6.4. Let us consider the equation

$$x'' + p(x)x' + ax = f(\sigma_t \omega),$$

where $p \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\Omega, \mathbb{R})$, $(\Omega, \mathbb{R}, \sigma)$ is a dynamical system with compact phase space Ω and a is a positive number. Denote by $y := x' + F(x)$, where $F(x) := \int_0^x p(s)ds$. Then we obtain the system

$$\begin{cases} x' = y - F(x) \\ y' = -ax + f(\sigma_t \omega). \end{cases} \tag{8}$$

Theorem 6.5. *Suppose the following conditions are held:*

1. *$p(x) \geq 0$ for all $x \in \mathbb{R}$;*
2. *there exist positive numbers r and k such that $p(x) \geq k$ for all $|x| \geq r$.*

Then the following statements hold:

(i) *the cocycle φ generated by (8) is dissipative and $V-$monotone;*
(ii) *the cocycle φ generated by (8) admits at least one continuous invariant section $\gamma : \Omega \to \mathbb{R}^2$;*
(iii) *If Ω is a compact minimal set containing only τ-periodic (quasi periodic, almost periodic, almost automorphic, recurrent) motions, then equation (6) admits at least one τ-periodic (quasi periodic, almost periodic, almost automorphic, recurrent) solution $\varphi(t, \gamma(y), y)$.*

Proof. Let $X := \Omega \times \mathbb{R}$ and $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ be the non-autonomous dynamical system generated by (8). We define the function $V : X \to \mathbb{R}^+$ by the equality

$$V(x, y) := y^2 - yF(x) + \frac{1}{2}F^2(x) + ax^2.$$

Then

$$\frac{d}{dt}V(\pi^t((x, \omega), (y, \omega))|_{t=0} = -p(x)[y - F(x)]^2 - axF(x) + (2y - F(x))f(\omega).$$

According to [16] (see the proof of Theorem 12.1.2), there exists $R > 0$ such that $\frac{d}{dt}V(\pi^t((x, \omega), (y, \omega))|_{t=0} < 0$ for all $x^2 + y^2 \geq R^2$ and $V(\omega, x, y) \to +\infty$ as $x^2 + y^2 \to +\infty$. In view of [**?**, Chapt.5] the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ is dissipative.

Let $V : E \times E \to \mathbb{R}^+$ be the function defined by the equality $V(u_1, u_2) := \langle u_1 - u_2), u_1 - u_2\rangle$. Then

$$\frac{d}{dt}V(\varphi(t, u_1, \omega), \varphi(t, u_2, \omega)) = -(x_1(t) - x_2(t))[F(x_1(t)) - F(x_2(t))] \leq 0$$

for all $t \in \mathbb{R}$, where $x_i(t) = pr_1\varphi(t, u_i, \omega)$ $(i = 1, 2)$, and, consequently,

$$V(\varphi(t, u_1, \omega), \varphi(t, u_2, \omega)) \leq V(u_1, u_2)$$

for all $t \in \mathbb{R}^+$. To finish the proof it is sufficient to refer to Theorems 4.10, 5.11 and Lemma 4.12. The theorem is proved. □

6.2. **Caratheodory's differential equations.** Let us consider now equation $x' = f(t, x)$ with the right hand side f satisfying the conditions of Caratheodory (see, for example,[20]). The space of all Caratheodory's functions we denote by $\mathfrak{C}(\mathbb{R} \times E, E)$. Topology on this space is defined by the family of semi-norms (see [20])

$$d_{k,m}(f) := \int_{-k}^{k} \max_{|x|\leq m} |f(t, x)|dt.$$

This space is metrizable, and on $\mathfrak{C}(\mathbb{R} \times E, E)$ there can be defined a dynamical system of translations $(\mathfrak{C}(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$.

We consider the equation

$$\frac{dx}{dt} = f(t, x), \tag{9}$$

where $f \in \mathfrak{C}(\mathbb{R} \times E, E)$, and the family of equations

$$\frac{dx}{dt} = g(t, x), \tag{10}$$

where $g \in H(f) := \overline{\{f_\tau \mid \tau \in \mathbb{R}\ \}}$, and f_τ is a $\tau-$translation of the function f w.r.t. the variable t, i.e. $f_\tau(t,x) := f(t+\tau,x)$ for all $t \in \mathbb{R}$ and $x \in E$, and by bar there is denoted the closure in the space $\mathfrak{C}(\mathbb{R} \times E, E)$. Denote by $\varphi(t,x,g)$ the solution of equation (10) with the initial condition $\varphi(0,g,x) = x$. Then φ is a cocycle on E (see, for example [20]) with the base $H(f)$. Hence, we may apply the general results from sections 1-4 to the cocycle φ generated by equation (9) with a Caratheodory's right hand side and will obtain some results for this type of equations.

For instance, the following assertion holds.

Theorem 6.6. *Let $f \in \mathfrak{C}(\mathbb{R} \times E, E)$ be an almost periodic function in $t \in \mathbb{R}$ (in the sense of Stepanoff* [14]*) uniformly w.r.t. x on compacts from E, i.e. for every $\varepsilon > 0$ and compact $K \subset E$ the set*

$$\mathfrak{T}(\varepsilon, f, K) := \{\tau \in \mathbb{R} \mid \int_0^1 \max_{x \in K} |f(t+\tau+s,x) - f(t+s,x)| ds < \varepsilon\ \}$$

is relatively dense on $\mathbb{R}$. Suppose that

1. *$\langle f(t,x_1) - f(t,x_2), x_1 - x_2 \rangle \leq 0$ for all $t \in \mathbb{R}$ and $x_1, x_2 \in E$;*
2. *there exists a positive constant r and a function $c : [r,+\infty) \to (0,+\infty)$ such that $\langle f(t,u), u \rangle \leq -c(|u|)$ for all $|u| > r$.*

Then on E equation (9) generates a cocycle φ which is dissipative and equation (9) admits at least one stationary ($\tau-$periodic, quasi-periodic, almost periodic) solution, if the function $f \in \mathfrak{C}(\mathbb{R} \times E, E)$ is stationary ($\tau-$periodic, quasi-periodic, almost periodic) in $t \in \mathbb{R}$ uniformly w.r.t. x on compacts from E.

Proof. This theorem is proved using the same arguments that we used in the proof of Theorem 6.2. □

6.3. **ODEs with impulse.** Let $\{t_k\}_{k \in \mathbb{Z}}$ be a two-sided sequence of real numbers, $p : \mathbb{R} \to \mathbb{R}^n$ be a continuously differentiable on every interval (t_k, t_{k+1}) function, continuous to the right in every point $t = t_k$, bounded on $\mathbb{R}$, almost periodic in the sense of Stepanoff and

$$p'(t) = \sum_{k \in \mathbb{Z}} s_k \delta_{t_k},$$

where $s_k := p(t_k + 0) - p(t_k - 0)$.

Consider the equation with impulse

$$\frac{dx}{dt} = f(t,x) + \sum_{k \in \mathbb{Z}} s_k \delta_{t_k}$$

or, what is equivalent,

$$\frac{dx}{dt} = f(t,x) + p'(t) \tag{11}$$

and parallely let us consider the family of equations

$$\frac{dx}{dt} = g(t,x) + q'(t), \tag{12}$$

where $(g,q) \in H(f,p) := \overline{\{(f_\tau, p_\tau) | \tau \in \mathbb{R}\}}$ and by bar we denote the closure in the product-space $\mathfrak{C}(\mathbb{R} \times E, E) \times \mathfrak{C}(\mathbb{R}, E)$.

Denote by $\varphi(t,x,g,q)$ the unique solution of the equation (12) (see [10] and [18]) satisfying the initial condition $\varphi(0,x,g,q)=x$. This solution is continuous on every interval (t_k,t_{k+1}) and continuous to the right in every point $t=t_k$ (see [10] and [18]).

By the transformation

$$x := y + q(t) \tag{13}$$

we can bring equation (12) to the equation

$$\frac{dy}{dt} = g(t, y + q(t)). \tag{14}$$

Theorem 6.7. *Let $f \in C(\mathbb{R}\times E, E)$ be a Bohr's almost periodic function in $t \in \mathbb{R}$ uniformly with respect to x on every compact from E and $p \in \mathfrak{C}(\mathbb{R}, E)$ be a Stepanoff's almost periodic function. Suppose that $\langle f(t,x_1)-f(t,x_2), x_1-x_2\rangle \le 0$ for all $t\in\mathbb{R}$, $x_1,x_2\in E$ and that there exist positive numbers α, L_1, L_2 and r such that*

$$\langle f(t,x),x\rangle \le -L_1|x|^{\alpha+1} \quad \text{and} \quad |f(t,x)| \le L_2|x|^{\alpha}$$

for all $t\in\mathbb{R}$ and $|x|>r$ $(x\in E)$.

Then the cocycle generated by equation (11) is dissipative and equation (11) admits at least one stationary (τ - periodic, quasi-periodic, almost periodic in the sense of Stepanoff) solution, if the function $(f,p)\in \mathfrak{C}(\mathbb{R}\times E,E)\times\mathfrak{C}(\mathbb{R},E)$ is stationary (τ - periodic, quasi-periodic, almost periodic in $t\in\mathbb{R}$) uniformly w.r.t. x on compacts from E.

Proof. Let $\varphi(t,x,g,q)$ be the cocycle generated by the family of equations (12) and $\tilde{\varphi}(t,y,g,q)$ be the cocycle generated by the family of equations (14). Then we have the following equality

$$\varphi(t,x,g,q) = q(t) + \tilde{\varphi}(t, x-q(0), g, q)). \tag{15}$$

We will show that it is possible to apply Theorem6.6 to the equation

$$\frac{dy}{dt} = f(t, y+p(t)).$$

Really,

$$\langle f(t,y_1+p(t)) - f(t,y_2+p(t)), y_1-y_2\rangle$$

for all $t\in\mathbb{R}$ and $y_1,y_2\in\mathbb{R}^n$, and

$$\langle f(t,y+p(t)),y\rangle = \langle f(t,y+p(t)), y+p(t)\rangle - \langle f(t,y+p(t)),p(t)\rangle \le \tag{16}$$

$$-L_1|y+p(t)|^{\alpha+1} + L_2\|p\||y+p(t)|^{\alpha}$$

for all $t\in\mathbb{R}$ and $|y+p(t)|>r$, where $\|p\| := \sup\{|p(t)| \mid t\in\mathbb{R}\ \}$. Taking into account the fact that the function p is bounded on $\mathbb{R}$, from (16) we obtain the existence of positive numbers R (that are sufficiently large) and L_1^0, L_2^0 such that

$$\langle f(t,y+p(t)),y\rangle \le -L_1^0|y|^{\alpha+1} \text{ and } \langle f(t,y+p(t))\rangle \le L_2^0|y|^{\alpha}$$

for all $t\in\mathbb{R}$ and $|y|>R$. To finish the proof of the theorem it is sufficient to apply Theorem 6.6 and take into consideration the relations (13) and (15). The theorem is proved. □

6.4. **Difference equations.**

Example 6.8. Consider the equation

$$u_{n+1} = f(n, u_n) \tag{17}$$

where $f \in C(\mathbb{Z}\times E, E)$; here $C(\mathbb{Z}\times E, E)$ is the space of all continuous functions $\mathbb{Z}\times E \to E$) equipped with a compact-open topology. This topology can be metrizable. For example, by the equality

$$d(f_1, f_2) := \sum_{1}^{+\infty} \frac{1}{2^n} \frac{d_n(f_1, d_2)}{1 + d_n(f_1, d_2)},$$

where $d_n(f_1, d_2) := \max\{\rho(f_1(k, u), f_2(k, u)) \mid k \in [-n, n],\ |u| \leq n\}$, there is defined a distance on $C(\mathbb{Z} \times E, E)$ which generates the topology of point-wise convergence with respect to $n \in \mathbb{Z}$ uniformly with respect to u on every compact from E.

Along with equation (17), we will consider the H-class of equation (17)

$$v_{n+1} = g(n, v_n) \quad (g \in H(f)), \tag{18}$$

where $H(f) = \overline{\{f_m \mid m \in \mathbb{Z}\}}$ and the over bar denotes the closure in $C(\mathbb{Z} \times E, E)$, and $f_m(n, u) = f(n+m, u)$ for all $n \in \mathbb{Z}$ and $u \in E$. Denote by $(C(\mathbb{Z} \times E, E), \mathbb{Z}, \sigma)$ a dynamical system of translations. Here $\sigma(m, g) := g_m$ for all $m \in \mathbb{Z}$ and $g \in C(\mathbb{Z} \times E, E)$.

Let Ω be the hull $H(f)$ of a given function $f \in C(\mathbb{Z}\times E, E)$ and denote the restriction of $(C(\mathbb{Z} \times E, E), \mathbb{R}, \sigma)$ on Ω by $(\Omega, \mathbb{Z}, \sigma)$. Let $F : E \times \Omega \to E$ be a continuous map defined by $F(u, g) = g(0, u)$ for $g \in \Omega$ and $u \in E$. Then equation (18) can be rewritten in this form:

$$u_{n+1} = F(u_n, \sigma^n \omega),$$

where $\omega := g$ and $\sigma^n \omega := g_n$.

Definition 6.9. *A function $f \in C(\mathbb{Z}\times E, E)$ is said to be periodic (almost periodic, recurrent), if $f \in C(\mathbb{Z} \times E, E)$ is a periodic (almost periodic, recurrent) point of the dynamical system of translations $(C(\mathbb{Z} \times E, E), \mathbb{Z}, \sigma)$.*

If the function $f \in C(\mathbb{Z}\times E, E)$ is periodic (almost periodic, recurrent), then the set $\Omega := H(f)$ is the compact minimal set of the dynamical system $(C(\mathbb{Z} \times E, E), \mathbb{Z}, \sigma)$ consisting of periodic (almost periodic, recurrent) points.

Theorem 6.10. *Let the function $f \in C(\mathbb{Z} \times E, E)$ be periodic (almost periodic,almost automorphic, recurrent) w.r.t. $n \in \mathbb{Z}$ uniformly w.r.t. u on compacts from E. If $|f(n, u_1) - f(n, u_2)| \leq |u_1 - u_2|$ for all $n \in \mathbb{Z}$ and $u_1, u_2 \in E$ and equation (17) admits a bounded on $\mathbb{Z}_+$ solution, then it admits also at least one periodic (almost periodic,almost automorphic, recurrent) solution.*

Proof. Note that under the condition of the theorem the cocycle φ generated by equation (17) satisfies the following inequality

$$|\varphi(n, u_1, g) - \varphi(n, u_2, g)| \leq |u_1 - u_2|$$

for all $u_1, u_2 \in E$, $n \in \mathbb{Z}_+$ and $g \in H(f)$, i.e. the cocycle $\langle E, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$ is V-monotone, where $V(u_1, u_2) := |u_1 - v_2|$. Now to finish the proof of the theorem it is sufficient to apply Theorems 4.10 and Lemma 4.12. $\square$

References

[1] Borisovich Yu. G., Gel'man B. D., Myshkis A. D. and Obuhovskii V. V., *Introduction to the theory of multivalued mappings. Multivalued mappings.* Voronezhskii Gosudarstvennyi Universitet, Voronezh, 1986.

[2] Borisovich Yu. G., Gel'man B. D., Myshkis A. D. and Obuhovskii V. V., Multivalued mappings. *Mathematical Analysis*, vol.19, pp.127-230, 232,

[3] Bronsteyn I. U., Akad. Nauk SSSR, Vsesoyuz.Inst.Nauchn. i Techn. Informatsii, Moscow, 1982. *Extensions of Minimal Transformation Group.* Noordhoff, 1979.

[4] Cheban D. N., *Asymptotic Almost Periodic Solutions of differential Equations.* University Press of USM, Chişinău, 2002.

[5] Cheban D. N., *Global Attractors of Non-autonomous Dissipative Dynamical systems.* World Scientific Pulishing Co, 2003(to appear).

[6] Cheresiz V. M., Almost Periodic Solutions of Nonlinear Systems, *Soviet Math. Dokl.***6**,1965, pp.1446-1449.

[7] Cheresiz V. M., V-Monotone Systems and Almost Periodic Solutions. *Sibirskii Matematicheskii Zhurnal.* 1972, v.13, No.4, pp.921–932.

[8] Cheresiz V. M., Uniformly V-Monotone Systems. Almost Periodic Solutions. *Sibirskii Matematicheskii Zhurnal.* 1972, v.13, No.5, pp.11107–1123.

[9] Coddington E. A. and Levinson N., *Theory of Ordinary Differential Equations.* McGraw Hill, New York, 1955.

[10] Halanay A. and Wexler D., *Teoria Calitativă a Sistemelor cu Impulsuri.* Bucureşti, 1968.

[11] Hartman P., *Ordinary Differential Equations.* Birkhauser, Boston–Basel–Stuttgart, 1982.

[12] Kantorovich L. V. and Akilov G. P., *Functional Analysis.* Third edition. Nauka, Moskow, 1984.

[13] Kelley J., *General Topology.* D. Van Norstand Company Inc., Toronto–New York–London, 1957.

[14] Levitan B. M. and Zhikov V. V., Almost Periodic Functions and Differential Equations.*Cambridge Univ. Press, Cambridge*, 1982.

[15] Opial Z., Sur les solutions presque-pèriodiques des èquations differentielles du premier et du second ordre, *An. Polon. Math.*, **7**, 1959, pp.51-61.

[16] Pliss V. A. *Nonlocal Problems in the Theory of Oscilations.* Academic Press, 1966.

[17] Rudin U., *Functional Analysis.* Mir, Moscow, 1975.

[18] Samoilenko A. M. and Perestiuk N. A., *Impulsive Differential Equations.* World Scientific Publishing Co., Inc., River Edge, NY, 1995.

[19] Sell G. R., W. Shen and Y. Yi, Topological Dynamics and Differential Equations, *Contemporary Mathematics*, **215**, 1998, pp.279-297.

[20] Sell G. R., Lectures on Topological Dynamics and Differential Equations, **vol.2** of *Van Nostrand Reinhold math. studies.* Van Nostrand–Reinbold, London, 1971.

[21] Sacker R. J. and Sell G. R., *Lifting Properties in Skew-Product Flows with Applications to Differential Equations*, Memoirs Amer. Math. Soc., **11**, 1977.

[22] Sell G. R. and You Y., *Dynamics of Evolutionary Equations.* Springer-Verlag, New York, 2002.

[23] Shcherbakov B. A., *Poisson Stability of Motions of Dynamical Systems and Solutions f Differential Equations.* Ştiinţa, Chişinău, 1985.

[24] Zhikov V. V., Monotonicity in the Theory of Almost Periodic Solutions of Nonlinear Operator Equations. *Math. USSR-Sb.***19**,No.2, 1973, pp.209-223.

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