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Global attractor in Solow growth model with differential savings and endogenic labor force growth.

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Abstract

In this paper we study the dynamics of a discrete triangular system T in capital per capita and population growth representing the neoclassical growth model with CES production function and differential savings, under the assumption that the labor force growth rate is endogenous and described by a generic iterative scheme having a unique positive globally stable equilibrium $\bar{n}$. The study herewith presented aims at confirming the existence of a compact global attractor for system T along the invariant line $\bar{n}$. Consequently asymptotic dynamics of growth models with constant population growth rate can be related to those with non-constant population growth if the steady state rate is globally stable. Furthermore we prove that the system exhibits cycles or even chaotic dynamics patterns if shareholders save more than workers, when the elasticity of substitution between production factors drops below one (so that capital income declines). The analytical results are supplemented by numerical simulations.

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keywords: Compact global attractor, chaotic dynamics, endogenic population growth.

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1 Introduction

Dynamic economic growth models have often considered the standard, one-sector neoclassical model by Ramsey (1928) or the Solow-Swan model (see Solow, 1956, and Swan, 1956). Both these dynamic models show that the system monotonically converges to the steady state (i.e. the capital per capita equilibrium) so neither cycles nor complex dynamics can be observed (see also Dechert, 1984). However, while Ramsey's assumption on savings behavior corresponds to maximizing the discounted sum of utility of a representative consumer who lives infinitely, in the Solow-Swan model, constant average propensity to save is assumed.

While considering one-sector growth models, other authors (i.e. Kaldor, 1956 and 1957, Pasinetti, 1962 and Samuelson & Modigliani, 1966) studied the following issue: whether the different saving propensity of two groups (labor and capital) as first proposed by Chiang (1973), might influence the final dynamic of the system. The question of differential savings between groups of agents was originally posed within the Harrod-Domar model of fixed portion (Harrod 1939). Stiglitz (1969) took Solow's model to another level by analyzing how different savers' wealth and income evolved. In his model each agent follows his or her private decision role and the economy approaches a balanced growth solution. Obviously different but constant saving propensities make the aggregate saving propensity non-constant and dependent on income distribution so that multiple and unstable equilibria can occur. However, qualitative dynamics are still simple.

More recently, in Bohm & Kaas (2000) the role of differential simple savings behavior as proposed by Kaldor (1956) and its distribution effects with regard to stability of stationary steady states has been investigated for the discrete-time Solow growth model. The authors show how instability and topological chaos can be generated in this kind of model.

In all the suggested works, growth models with constant labor force growth rate have been investigated; however one implication of a constant population growth rate is that population grows exponentially, which is clearly unrealistic. For this reason, many authors consider it more realistic to describe the population growth using growth functions different from the exponential growth one. Verhulst (see Schtickzelle and Verhulst 1981) was the first who proposed to model population growth with the logistic equation. Other authors made the same choice: Accinelli and Brida (2005) analyze the neo-classical Solow model with growth of population described by a generalized logistic equation (Richard's law), Faria (2004) studies the Ramsey model with logistic growth.

A further development is in Brianzoni Mammama Michetti (2006) where a similar model to that studied in Bohm and Kaas (2000) has been investigated under the assumptions of CES production function and the labor force growth rate not being constant (in particular a model for density dependent population growth described by the Beverton-Holt equation[1] has been considered[2]). The model obtained is then a bidimensional autonomous dynamic system; the authors proved that multiple equlibria are likely to emerge, and they provided conditions on the parameters. They also showed how fluctuations and complicated dynamics may arise when the elasticity of substitution between the two production factors drops below one.

In the present work a triangular system $(T, \mathbb{R}^2_+)$ is proposed to model the evolution of capital accumulation and population growth rate in a discrete-time Solow growth model. More precisely, the study herewith presented assumes differential savings as in Bohm and Kaas (2000),[3] CES production function as in Brianzoni S., Mammana C. and Michetti E. (2006) and, finally, a recurrence $f(n)$ describing population dynamics, having a unique globally asymptotically stable fixed point. In fact the less convincing element of the works above mentioned is that assigned functions (i.e. the logistic or the Beverton Holt equation) have been used to describe the population dynamic evolution, while, more realistically, the exact law governing population dynamics is not known in general but only its properties are noted (i.e. that a steady state growth rate will be reached in the long run).

According to such considerations and on the basis of the results already attained, we study the more realistic growth model characterized by a generic law for population dynamics. Consequently we generalize some results obtained in Brianzoni S., Mammana C. and Michetti E. (2006) which have been extended to each iterative scheme for population dynamics which is differentiable and having a unique globally stable fixed point.

The analysis of the dynamics characterizing our triangular system is not a simple thing, as most of the results cannot be expressed in analytical form; this obstacle has been overcome by reducing the two-dimensional system to a one-dimensional map that represents a limiting form. In particular it is proved that a global attractor for system T does exist and it can be searched using the limiting map. Furthermore, since the limiting map is the restriction

[1]See Beverton and Holt (1957).

[2]The Beverton-Holt model in discrete time is equivalent to logistic model.

[3]We differentiate saving rates across the functional distribution of income, consequently it is implicitly assumed that workers and capitalists are different individuals. Furthermore our model does not assume private agents engage in explicit optimization decisions when undertaking their saving-consumption decisions.

of system T to the line $n = \bar{n}$, being $\bar{n}$ the equilibrium population growth rate, in our work we are able to generalize the results obtained in growth model with constant population growth rate to the ones with endogenic population growth rate, if the process converges to a global attracting steady state.

The study of the dynamics exhibited by our model shows how the capital accumulation dynamics (and hence the growth patterns) are more and more complex as the elasticity of substitution between the two production factors is low enough, and that the map is chaotic if such elasticity tends to zero. The Leontief production function is just the extreme example of this observation.[4] The results obtained aim at confirming that the production function's elasticity of substitution plays a central role in the creation and propagation of complicated dynamics as in models with explicitly dynamic optimizing behavior by the private agents.[5]

Furthermore an important role is played by the two propensities to save: i. e. if workers save more than shareholders only simple dynamics are possible while, in the opposite case, conditions for topological chaos to be owned are pursued.

The paper is organized as follows. In section 2 we introduce the model. In section 3 we prove theorems concerning global dynamics and the existence of a compact global attractor. In particular, we demonstrate that the global attractor belongs to the invariant line $n = \bar{n}$ so that T admits a one-dimensional limiting form informing on the asymptotic dynamics of the system. In section 4 we prove that the limiting map is chaotic for some values of the parameter and we introduce some numerical simulations showing that complex dynamics can be observed. Section 5 concludes our paper.

2 The model

Recall the following system $(T, \mathbb{R}^2_+)$ describing capital per capita (k) and population growth rate (n) dynamics[6] of the model studied in Brianzoni S., Mammana C. and Michetti E. (2006), where $T = (T_1, T_2)$

[4]While considering production function with fixed portion we reach a similar conclusions about cyclic behavior to that by Nashimura and Yano (1995) for the case of one sector model.

[5]The survey paper by Becker (2006) covers many of these models.

[6]Labor force growth rate is described by the Beverton-Holt equation representing a model for density dependent population growth (see Beverton and Holt, 1957) that has been largely studied in Cushing and Henson (2001, 2002).

$$T_1(n) = \frac{rh}{h + (r-1)n} n$$

$$T_2(n,k) = \frac{1}{1+n} \left[(1-\delta)k + (k^\rho + 1)^{\frac{1-\rho}{\rho}} (s_w + s_r k^\rho) \right]$$

$\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders respectively $\rho \in (-\infty, 1)$, $\rho \neq 0$ is a parameter related to the elasticity of substitution between labor and capital,[7] $h > 0$ is the carrying capacity and $r > 1$ is the inherent growth rate.

The previous system has been obtained while considering the standard, neoclassical one-sector growth model where the two types of agents, workers and shareholders, have different but constant saving rates as in Bohm & Kaas (2000) and where production function f, mapping capital per worker k into output per worker y, is of the CES type, that is

$$y = F(k) = (1 + k^\rho)^{\frac{1}{\rho}}. \tag{1}$$

The present work is dedicated to the study of a generalization of the model just described, since the Beverton–Holt equation $T_1(n)$ is replaced by a generic map for population dynamics, as stated in the following definition.

Definition 2.1. $T_1(n) := f(n)$ *is a continuous differentiable map having a single positive fixed point* $n = \bar{n}$ *globally asymptotically stable.*

Here we model the evolution of the population growth rate n by an iterative scheme $n_{t+1} = f(n_t)$ such that $\forall n_0 \in \mathbb{R}_+$ the steady state $n = \bar{n}$ is approached, i. e. $\lim_{t \to +\infty} f^t(n_0) = \bar{n}$.

Such an iteration scheme represents a generalization of many equations able to describe population dynamics often considered in economic application (i. e. the Beverton-Holt equation or the logistic function as previously underlined). Furthermore it is quite realistic to allow population growth rate varying over time.

By considering function f, the final triangular map is given by

$$T := \begin{cases} T_1(n) = f(n) \\ \\ T_2(n,k) = g(n,k) \end{cases} \tag{2}$$

[7]Remember that the elasticity of substitution between the two production factors is given by $\frac{1}{1-\rho}$.

where $g(n,k) := \frac{1}{1+n}\left[(1-\delta)k + (k^\rho+1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho)\right]$. If $\rho < 0$, we assume $g(n,0) := 0$ $\forall n \in \mathbb{R}_+$ for g being continuous. Map T is well defined. In what follows we investigate qualitative and quantitative dynamic properties of system (2).

3 Global dynamics

In this section we give conditions on the parameters for T having a global attractor. Furthermore we will prove that T has a one-dimensional limiting form having the same asymptotic behavior of T. Consequently the qualitative asymptotic dynamics of T can be studied while considering its simpler one-dimensional form.

First we determine the invariant sets of map T in order to discuss their global stability. Remember the following definition.

Definition 3.1. *Recall that a subset $E \subseteq \mathbb{R}^2_+$ is invariant (positively invariant, negatively invariant) if $T^t(E) = E$ ($T^t(E) \subseteq E$, $E \subseteq T^t(E)$), $\forall t \in \mathbb{Z}_+$.*

In the following statement we prove that T admits both an invariant and a positively invariant set.

Lemma 3.2. *The set $E_0 = \{(\bar{n}, k) : k \in \mathbb{R}_+\}$ is invariant and the set $E = \{(n,k) : |n - \bar{n}| \leq \epsilon, k \in \mathbb{R}_+\}$ is positively invariant for the mapping T.*

Proof. E_0 is invariant as mapping T is triangular and $n = \bar{n}$ is a fixed point for the one-dimensional map $T_1(n)$. Set E is positively invariant since $\forall (n_0,k_0) \in E, T^t(n_0,k_0) = (f^t(n_0), g^t(n_0,k_0)) \in E, \forall t \in \mathbb{Z}_+$, as $|n_0 - \bar{n}| \leq \epsilon$ and $n = \bar{n}$ is stable for g. □

From this lemma it follows that any initial condition (n_0, k_0) belonging to set E has an orbit which is bounded inside this set; furthermore the half-line $n = \bar{n}$ is mapped into itself by forward and backward iterations of T.

Consider the following definition of a global attractor.

Definition 3.3. *A nonempty compact set $C \subset \mathbb{R}^2_+$ is the global attractor of the dynamical system $(T, \mathbb{R}^2_+)$ if the following conditions are fulfilled:*

a. C is invariant with respect to $(T, \mathbb{R}^2_+)$;

b. C attracts all the bounded subsets from $\mathbb{R}^2_+$. [8]

[8]See Cheban D. (2004).

We first consider system T with $\rho \in (0,1)$ and we prove the following Theorem about the existence of a global attractor.

Theorem 3.4. *If $\rho \in (0,1)$ and $s_r < \delta$ then dynamical system $(T, \mathbb{R}^2_+)$ admits a compact global attractor.*

Proof. Consider map $g(n,k)$ as expressed in the following manner:

$$g(n,k) = \frac{1}{1+n}\left[(1-\delta)k + (k^\rho+1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho)\right] =$$

$$= \frac{1}{1+n}\left[(1-\delta+s_r)k + (k^\rho+1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho) - s_r k\right] =$$

$$= \frac{1}{1+n}\left[(1-\delta+s_r)k + h(k)k\right]$$

where $h(k) := \frac{(k^\rho+1)^{\frac{1}{\rho}}}{k}\frac{(s_w+s_r k^\rho)}{k^\rho+1} - s_r$. Since ρ is positive, $\lim_{k\to+\infty} h(k) = 0$ that is $\forall \gamma > 0, \exists M > 0$ such that $|h(k)| < \gamma$, $\forall k > M$. Consequently

$$g(n,k) < \frac{1}{1+n}[(1-\delta+s_r+\gamma)k] < (1-\delta+s_r+\gamma)k$$

$\forall k > M$. Define $A := [\bar{n}-\epsilon, \bar{n}+\epsilon] \times [0, M]$ then the trajectory $\{T^t(n_0,k_0) : t \in \mathbb{Z}_+\}$ at least one time intersects the compact A, $\forall (n_0,k_0) \in \mathbb{R}^2_+$ with $k_0 > M$. In fact, if we suppose that this statement is false, then there exists a point $(n_0,k_0) \in \mathbb{R}^2_+ \setminus A$ such that $T^t(n_0,k_0) \in \mathbb{R}^2_+ \setminus A$, $\forall t \in \mathbb{Z}_+$. Since $\lim_{t\to+\infty} f^t(n_0) = \bar{n}$, being $\bar{n}$ globally stable for map f, then it must be

$$k_t = g^t(n_0,k_0) > M. \tag{3}$$

Assume $\gamma < \delta - s_r$ then

$$k_t = g^t(n_0,k_0) < (1-\delta+s_r+\gamma)^t k_0 \longrightarrow 0 \; as \; t \to +\infty. \tag{4}$$

Equations (3) and (4) are contradictory. The contradiction obtained proves that set A is attracting for system T.

To prove that A is a positively invariant set, assume $k_0 < M$ and $\gamma < \delta - s_r$. Consequently, from (4), we have $k_t < k_0, \forall t \in \mathbb{Z}_+$ so that $k_t < M, \forall t \in \mathbb{Z}_+$. Furthermore, since lemma 3.2 holds, compact set A is positively invariant. Since A is a trapping region for T (i.e. a closed region positively invariant) it follows that $\Lambda = \cap_{t\geq 0} T^t(A)$ is the global attractor. $\square$

We consider now the case $\rho < 0$ and we prove the following Theorem.

Theorem 3.5. *If $\rho < 0$ then the dynamic system $(T, \mathbb{R}^2_+)$ admits a compact global attractor.*

Proof. Consider map $g(n,k) = \frac{1}{1+n}[(1-\delta)k + j(k)]$ where $j(k) := (k^\rho + 1)^{\frac{1-\rho}{\rho}}(s_w + s_r k^\rho)$. $\lim_{k\to+\infty} j(k) = s_w$, being $\rho < 0$, so that $\forall \eta > 0, \exists L > 0$ such that $|j(k) - s_w| < \eta$, $\forall k > L$. Similarly to the previous proof we obtain:

$$g(n,k) < \frac{1}{1+n}[(1-\delta)k + s_w + \eta], \ \ \forall\, k > L$$

and

$$k_t = g^t(n_0, k_0) < (1-\delta)^t k_0 + (s_w + \eta)\sum_{i=0}^{t-1}(1-\delta)^i \longrightarrow \frac{s_w + \eta}{1-\delta}$$

as $t \to +\infty$. For the arbitrary of η we can conclude in a similar manner to that of the previous theorem. □

Since T is a continuous system which maps the trapping set $K := [\bar{n} - \epsilon, \bar{n} + \epsilon] \times [0, N]$ [9] into itself, then T has at least one fixed point in K in cases $\rho < 0$ and $\rho \in (0,1) \cap \{s_r < \delta\}$ that will be considered from now on.

According to the previous theorems, we can conclude that the global asymptotic dynamics exhibited by system T must be investigated in the rectangle K. Furthermore, while considering that E_0 and K are respectively invariant and positively invariant for T, we can deduce that also the segment $S = E_0 \cap K$ is positively invariant.

The global attractor $\Lambda = \cap_{t\geq 0} T^t(K)$ lies in the rectangle $K = [\bar{n} - \epsilon, \bar{n} + \epsilon] \times [0, N]$, more stronger $\Lambda \subseteq S = \{(\bar{n}, k) : 0 \leq k \leq N\}$. In fact, if this statement is false, then the distance $\bar{d} := \max_{(n,k)\in A} |n - \bar{n}|$ will be strictly positive and an arbitrary small ϵ can be chosen such that $0 < \epsilon < \bar{d}$, so that $A \nsubseteq K$ in contradiction with theorems 3.4 and 3.5.

More precisely the asymptotic states for system T, as m-cycles, are related to m-cycles of the limiting form map defined as follows:

$$g_{\bar{n}}(k) := g(\bar{n}, k). \tag{5}$$

This can be intuitively justified on the basis that the sequences n_t are convergent to $\bar{n}$ as proved in following theorem.

[9] Being $N = M$ if $\rho < 0$ and $N = L$ if $\rho > 0$ as defined in the proofs of theorems 3.4 and 3.5.

Theorem 3.6. *If T has an m-period cycle, namely*

$$O_m = \{(n_0, k_0), (n_1, k_1), ..., (n_{m-1}, k_{m-1})\},$$

then $O_m \in S$.

Proof. Observe that if $i \in [0, m-1]$ exists such that $n_i = \bar{n}$, then $n_i = \bar{n}$ $\forall\, 0 \le i \le m-1$.

To prove the theorem, we suppose that $\bar{n} \neq n_0$. Since n_0 is a periodic point with period m, then $f^m(n_0) = n_0$ and $f^{tm}(n_0) = n_0$ $\forall\, t \in \mathbb{Z}_+$. Consequently $\lim_{t\to+\infty} f^{tm}(n_0) = n_0$, but this statement is contradictory with the global asymptotic stability of the fixed point $\bar{n}$ for f. Moreover, $k_i \le N$, $\forall\, 0 \le i \le m-1$ being K globally attracting and positively invariant. □

Remembering the previous result stated in theorem 3.6, we define two sequences

$$O_m = \{(\bar{n}, k_0), (\bar{n}, k_1), ..., (\bar{n}, k_{m-1})\} \tag{6}$$

and the correspondent

$$O'_m = \{(k_0), (k_1), ..., (k_{m-1})\} \tag{7}$$

and we consider $g_{\bar{n}}$ being the one-dimensional continuous map in $\mathbb{R}_+$ defined in (5). As a consequence of theorem 3.6 it is straightforward to prove that O_m is an m-cycle of period $m \ge 1$ of T if and only if O' is an m-cycle of period $m \ge 1$ of $g_{\bar{n}}$.

It is an important result since in order to find the periodic solution owned by T we can consider its one-dimensional limit form $g_{\bar{n}}$.

The question which arises is whether a similar result applies when considering the stability of periodic cycles. The Jacobian matrix of T is the triangular matrix:

$$DT(n,k) = \begin{pmatrix} f'(n) & 0 \\ \frac{\partial g}{\partial n}(n,k) & \frac{\partial g}{\partial k}(n,k) \end{pmatrix}. \tag{8}$$

As we know that f has a single fixed point $n = \bar{n}$ globally asymptotically stable we are interested in evaluating DT in points of that line.

Consider first O_m as in (6) with $m = 1$. We have

$$DT(\bar{n},k_0) = \begin{pmatrix} f'(\bar{n}) & 0 \\ \frac{\partial g}{\partial n}(\bar{n},k_0) & \frac{\partial g}{\partial k}(\bar{n},k_0) \end{pmatrix}. \tag{9}$$

where $(\bar{n}, k_0)$ is a fixed point of T. The eigenvalues are $\lambda_1 = f'(\bar{n})$, that is lesser than one in modulus, and $\lambda_2 = \frac{\partial g}{\partial k}(\bar{n}, k_0) = g'_{\bar{n}}(k_0)$. It follows that every fixed point of T is a stable node if and only if k_0 is an attracting fixed point of the one-dimensional map $g_{\bar{n}}$.

The same is true for an $m-$cycle with $m > 1$. In fact we know that (6) is a $m-$cycle of T if and only if (7) is a $m-$cycle of $g_{\bar{n}}$ and an $m-$cycle of T is attracting if and only if the m fixed points of the map T^m are attracting. Then we consider the Jacobian matrix of T^m: $DT^m = DT(\bar{n}, k_0) \cdot DT(\bar{n}, k_1) \cdot \ldots \cdot DT(\bar{n}, k_{m-1})$ whose eigenvalues are $\lambda_1^{(m)} = \prod_{i=0}^{m-1} g'_{\bar{n}}(k_i)$, $(i = 0 \ldots m-1)$ and $\lambda_2^{(m)} = [f'(\bar{n})]^m$. As theorems 3.4 and 3.5 hold, conclusions about the global stability can be easily given in the following Theorem.

Theorem 3.7. *O_m is an attracting m-cycle of period $m \geq 1$ of T if and only if O'_m is an attracting m-cycle of period $m \geq 1$ of $g_{\bar{n}}$.*

The theoretical study herewith proposed is interesting in that it allows conclusions to be drawn about growth models with endogenic population growth. In fact, if the population dynamics approach a globally stable fixed point $n = \bar{n}$, then models with constant population growth rate $n = \bar{n}$ have the same asymptotic dynamics of those with non-constant population growth having a globally stable equilibrium and, consequently, the strong assumption of constant labor force growth rate is not too restrictive when considering the long run economy.

4 Bifurcations and chaos.

Our previous considerations come from the identification of the restriction of T to the line $n = \bar{n}$ with the one-dimensional map

$$g_{\bar{n}}(k) = \frac{1}{1+\bar{n}} \left[(1-\delta)k + (k^\rho + 1)^{\frac{1-\rho}{\rho}} (s_w + s_r k^\rho) \right]. \tag{10}$$

The equivalence between asymptotic dynamics of T and those of $g_{\bar{n}}$ proved in section 3 is clearly useful in order to define which are the possible attracting sets of T.

For instance if $g_{\bar{n}}$ has a unique fixed point $\bar{k}$ in $\bar{S} := [0, N]$ locally attracting and no other invariant sets in $\bar{S}$ then every orbit of T converges to $\bar{k}$. The last statement gives a sufficient condition for the global convergence to the steady state of the growth model with endogenous population growth. Similarly if $g_{\bar{n}}$ has a unique attracting m-cycle in $\bar{S}$ then it is an attracting m-cycle of T. The same is true for any attracting set.

These results are useful in applications since they permit us to obtain complete understanding of the asymptotic behavior of the growth dynamics from the study of the one dimensional map $g_{\bar{n}}(k)$. Consequently, by knowing the geometrical properties of $g_{\bar{n}}$[10] we can draw conclusions about the invariant sets of T and their stability depending on all the parameters of the system. Furthermore the known results about local stability of the attractors of $g_{\bar{n}}$ can now be extended to more general consideration on the plane $\mathbb{R}^2_+$.

Precisely, for some values of the parameters $g_{\bar{n}}$ is strictly increasing, consequently its invariant sets are fixed points at most, they could be stable or unstable: the seconds separate the basins of attraction of the stable ones. We recall such cases.

Let $g_{\bar{n}}$ given by (10) and $\rho \in (0, 1)$. If $s_r < \delta$ then theorem 3.4 holds, moreover map $g_{\bar{n}}$ has one positive fixed point which is asymptotically stable. Taking into account the results herewith obtained, we conclude that all the trajectories approach the equilibrium, not only the ones starting from some initial conditions (i. e. from a neighborhood of the steady state). Consequently $(\bar{n}, \bar{k})$ is the unique global attractor of T. No more complicated asymptotic dynamics are exhibited if ρ is positive.

This result is strictly related to the capital income monotonicity property.[11] In fact a sufficient condition for $(k_t F'(k_t))'$ being positive is that the production function's elasticity of substitution is greater than or equal to one. This well-known result applies to our model if $\rho \in (0, 1)$, in fact if capital income is monotonically increasing in capital, also $g_{\bar{n}}$ is so, then cycles and chaos cannot appear.[12] Obviously if $\rho < 0$ (and the elasticity of substitution is lesser than one), capital income monotonicity may not be verified, and the stationary equilibrium could fail to be unstable and complicated dynamics may arise (see Becker and Foias 1998).

Let $\rho < 0$. In the cases $\{\delta + \bar{n} < s_r < sw\} \cap \{\rho \geq \frac{s_r}{s_r - s_w}\}$ and $\{s_r > \max(s_w, \delta + \bar{n})\} \cap \{\rho \geq \frac{-s_w}{s_r - s_w}\}$ map $g_{\bar{n}}$ has a repelling fixed point at the origin and another one $\bar{k} > 0$ that is locally asymptotically stable. Remembering theorem 3.5 we can conclude that the positive fixed point attracts any trajectory starting from an initial condition with $k_0 \neq 0$ (see figure 1, panel (a)) and the attractor Λ consists of the two fixed points.

Moreover for $\{s_r < s_w\} \cap \{\rho < \frac{s_r}{s_r - s_w}\}$, map $g_{\bar{n}}$ may have up to three fixed points: $k = 0$ and $0 < \bar{k}_1 < \bar{k}_2$. If they all exist, $k = 0$ and $\bar{k}_2$ are asymptotically stable. Obviously trajectories starting from an initial

[10] These properties have been studied in Brianzoni S., Mammana C. and Michetti E. (2006).

[11] According to this property if capital income is increasing in capital, no fluctuations are possible.

[12] Becker (2006) presents a detailed description of this relationship in heterogeneous agent models.

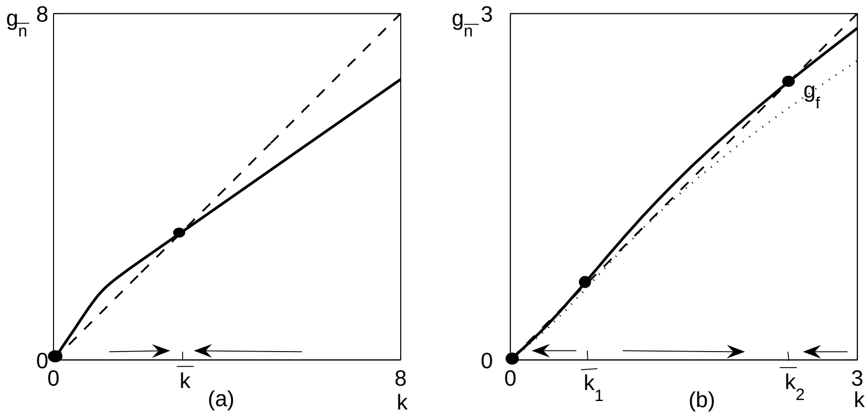

Figure 1: (a) Map $g_{\bar{n}}$ with $\delta = 0.29$, $h = 0.015$, $s_w = 0.9$, $s_r = 0.8$, $\rho = -7$. $\bar{k}$ is the asymptotically stable fixed point. (b) Map $g_{\bar{n}}$ with $\delta = 0.29$, $h = 0.015$, $s_w = 0.9$, $s_r = 0.2$, $\rho = -2$. Three fixed points exist where the repelling one $\bar{k}_1$ separates the basins of attractions of 0 and $\bar{k}_2$ respectively. Curve g_f and fold bifurcation.

condition $k_0 < \bar{k}_1$ will approach $k = 0$, while trajectories starting from an initial condition $k_0 > \bar{k}_1$ will approach $\bar{k}_2$. Consequently the attractor Λ consists of three pieces, the fixed points of T. Remember that in the previous section we stressed that the dynamics of map T are governed by map $g_{\bar{n}}$ only asymptotically so that it is a good approximation only in the long run. Note that for certain values of the parameters a fold bifurcation occur, see figure 1 panel (b).

The results proved in Brianzoni S., Mammana C. and Michetti E. (2006) enable us to conclude that if $\rho < 0$ and $s_r < s_w$ no complex dynamics can be exhibited. In fact no fluctuations are possible if the elasticity of substitution is lesser than one and workers save more than shareholders. This conclusion might appear to be in contrast with the possible failure of capital income monotonicity property being $\rho < 0$. Nevertheless it is especially in this case where the two different propensities to save play a role in our paper. In fact, if $s_r - s_w < 0$, map $g_{\bar{n}}$ is an ascending monotonic function independently of the elasticity of substitution's sign of magnitude.

In order to obtain cycles or chaos we have to consider $\rho < 0$ and $s_r > s_w$, such a case needs in fact more attention since $g'_{\bar{n}}$ sign may change. In this last case we can consider a further result, for example, if $\rho \in \left[-\frac{s_w}{s_r - s_w}, 0\right)$ function $g_{\bar{n}}$ is increasing so that only simple dynamics are exhibited, consequently in what follows we focus on the case $\rho < -\frac{s_w}{s_r - s_w}$.[13] Observe that a decline in capital income as capital increases represents a source of fluctuations in our model and that it is strictly related to the elasticity of production function: the elasticity of substitution must be pushed sufficiently below one in order for the non-monotonicity in capital income to be fulfilled and fluctuations to be exhibited.

Since map $g_{\bar{n}}$ is quite complicated to be studied analyticaly, in what follows we consider an approximation of such a map that permits us to obtain a further result about topological chaos for map $g_{\bar{n}}$. We use the following arguments.

Consider that the Leontief function $F^l = \min\{1, k\}$ is approximated by the family of concave production function

$$F_\rho(k) = (1 + k^\rho)^{\frac{1}{\rho}}, \quad \rho < 0.$$

If $k \neq 1$ it is straightforward to show that

$$\lim_{\rho \to -\infty} F_\rho(k) = F^l(k), \lim_{\rho \to -\infty} F'_\rho(k) = (F^l)'(k). \tag{11}$$

Consequently g^l given by

$$g^l := \begin{cases} g^l_1(k) = \frac{1}{1+\bar{n}} \left(1 - \delta + s_r\right) k, & \text{if } k < 1 \\ \\ g^l_2(k) = \frac{1}{1+\bar{n}} \left[(1-\delta)k + s_w\right], & \text{if } k > 1 \end{cases} \tag{12}$$

is the map for the approximation of $g_{\bar{n}}(k)$. Furthermore, because (11) holds, then $g_{\bar{n}}(k) \to g^l$, $\forall k \neq 1$ if $\rho \to -\infty$. Define $g^l_1(1) := \lim_{k \to 1^-} g^l$ and $g^l_2(1) := \lim_{k \to 1^+} g^l$.

Recall Brianzoni S., Mammana C. and Michetti E. (2006), where a proposition is proved stating sufficient condition for $g_{\bar{n}}$ being bimodal (that is it admits a maximum point k_M and a minimum point k_m with $k_M < k_m$) when ρ is small enough (i.e. $\forall \rho < \rho_1$).

In the following theorem we prove that the unique positive fixed point $\bar{k}$ of $g_{\bar{n}}$ belongs to the interval (k_M, k_m) if ρ is small enough.

Theorem 4.1. *Let $s_w < \delta + \bar{n} < s_r$ then $\bar{\rho} < 0$ does exist such that $k_M < \bar{k} < k_m$, $\forall \rho < \bar{\rho}$.*

[13] See again Brianzoni, Mammana and Michetti (2006).

Proof. It is easy to prove that a value ρ_2 exists such that $g'_{\bar{n}}(1) < 0 \; \forall \rho < \rho_2$, and consequently $k_M < 1 < k_m$.

Since $g_{\bar{n}}(k) \to g^l$, $\forall k \neq 1$ if $\rho \to -\infty$ then $\forall \epsilon > 0$, ρ_3 exists such that $-\epsilon < g_{\bar{n}}(k) - g^l(k) < \epsilon$, $\forall \rho < \rho_3$ and $k \neq 1$. Let $k = k_M$ then $g_1^l(k_M) - \epsilon < g_{\bar{n}}(k_M) < g_1^l(k_M) + \epsilon$. Since $k_M < g_1^l(k_M)$, being $\delta + \bar{n} < s_r$, then $k_M - \epsilon < g_{\bar{n}}(k_M)$. Let $k = k_m$ then $g_2^l(k_m) - \epsilon < g_{\bar{n}}(k_m) < g_2^l(k_m) + \epsilon$. Since $k_m > g_2^l(k_m)$, being $s_w < \delta + \bar{n}$, then $g_{\bar{n}}(k_m) < k_m + \epsilon$. Let $\bar{\rho} := \min\{\rho_1, \rho_2, \rho_3\}$ then for the arbitrary of ϵ the statement is proved. □

According to the previous theorem, we conclude that if ρ is small enough,[14] map $g_{\bar{n}}$ is a $Z_1 - Z_3 - Z_1$ map, where Z_i is an open interval the points of which have i distinct preimages.

On the other hand, g^l is of the kind $Z_1 - Z_2 - Z_1$; in this case Z_2 is the interval $(g_2^l(1), g_1^l(1))$ (absorbing segment), the first rank critical points are $g_2^l(1) = \frac{1-\delta+s_w}{1+\bar{n}}$ and $g_1^l(1) = \frac{1-\delta+s_r}{1+\bar{n}}$. Every initial condition $k_0 \neq 0$ belonging to Z_1 generates an iterated sequence which after a finite number of iterations penetrates inside Z_2 and tends toward an attractor located on this absorbing segment.

We now want to prove that parameter values exist such that the limiting map $g_{\bar{n}}$ is chaotic. Consider the Leontief production function F^l for which the time one map is (12), then the following considerations hold.[15]

First notice that if $s_w < \delta + \bar{n} < s_r$ then $g_2^l(1) < 1 < g_1^l(1)$. Furthermore $g_1^l(1) = g_2^l(1) + \frac{\triangle s}{1+\bar{n}}$, where $\triangle s = s_r - s_w$. Define $\gamma = 1 - g_2^l(1)$, so that for γ sufficiently small one has $1 - g_2^l(1)$ small and $g_1^l(1) - 1$ close to $\frac{\triangle s}{1+\bar{n}}$.

In particular, for $k_0 > 1$ close to one has (see figure 2 panel (a)):

$$g^l(k_0) < 1 < k_0 < (g^l)^3(k_0) < (g^l)^2(k_0).$$

Because of $\lim_{\rho \to -\infty} g_{\bar{n}}(k) = g^l(k) \; \forall k \neq 1$, it follows that k_0 exists such that:

$$g_{\bar{n}}(k_0) < k_0 < g_{\bar{n}}^3(k_0) < g_{\bar{n}}^2(k_0)$$

for ρ sufficiently small. Being $g_{\bar{n}}$ continuous then, for the well known theorem of Li and Yorke, there exists a cycle of every order $m = 1, 2, 3, ...$, which implies the existence of topological chaos,[16] as stated in the following theorem.

[14] That is the elasticity of substitution between the production factors is small enough.
[15] For further details see Bohm and Kaas (2000).
[16] See Day (1994).

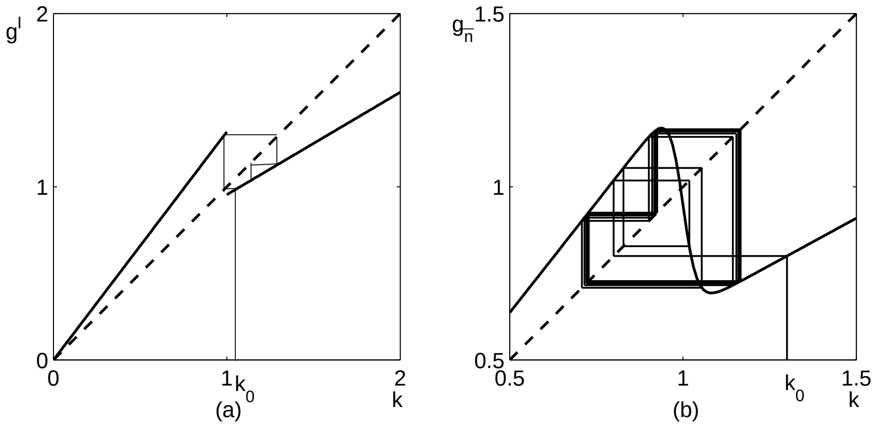

Figure 2: (a) Overshoot conditions with Leontief technology. (b) K-L staircase diagram of $g_{\bar{n}}$ being $\rho = -50$, $\delta = 0.4$, $\bar{n} = 0.1$, $s_w = 0.1$, $s_r = 0.8$. A three period cycle is exhibited.

Theorem 4.2. *Let $s_w < \delta + \bar{n} < s_r$, then $\tilde{\rho} < 0$ does exist such that $g_{\bar{n}}$ exhibits topological chaos $\forall \rho < \tilde{\rho}$.*

In the previous theorem, we found that a lower unbounded set does exist such that map $g_{\bar{n}}$ is topologically chaotic for all ρ in this set. It states a necessary condition for chaos to be observed in our model. It is straightforward to prove that theorem 4.2 holds if the elasticity of substitution falls sufficiently below one and the capital income monotonicity condition fails to hold (since ρ is small enough). The Leontief-fixed coefficient production case is the extreme example of the decline in the elasticity of substitution between two factors and hence of the failure of monotonicity property of capital income.

We now consider the Koneig-Lamerary staircase diagram of map $g_{\bar{n}}$ in figure 2 panel (b). The trajectory starting from a generic initial condition shows that a three period cycle is approached. Consequently, map $g_{\bar{n}}$ is topologically chaotic[17] according to what has been proved in the previous theorem.

Since the analytic form of the limiting map $g_{\bar{n}}$ is quite complicated, the dynamic behavior when varying one or more parameters of the map must be

[17]See Li and Yorke (1975).

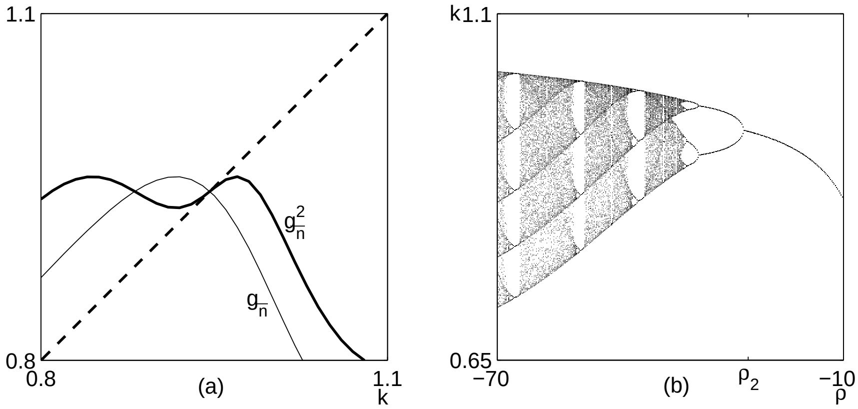

Figure 3: (a) Map $g_{\bar{n}}$ and $g^2_{\bar{n}}$: period doubling bifurcation at $\rho = \rho_2 \simeq -26.5$ with $\delta = 0.4$, $\bar{n} = 0.1$, $s_w = 0.1$, $s_r = 0.6$. (b) Bifurcation diagrams of map $g_{\bar{n}}$ with respect to ρ and the other parameters as in panel (a): more complex dynamics can be observed as ρ decreases.

analyzed numerically.

First, when studying the bifurcations of map $g_{\bar{n}}$ we consider the variations affecting coefficient ρ which represents the only parameter presented in the production function and related to the elasticity of substitution of the two factors. Furthermore, as proved in theorem 4.2, we expect to observe more complex dynamics when ρ is sufficiently small so that the elasticity of substitution between capital and labor is close to zero. We also consider parameter values such that $s_w < \delta + \bar{n} < s_r$ that is the hypotheses of theorem 4.2 are fullfilled.

Notice that as ρ decreases, into the second iteration of $g_{\bar{n}}$ more fixed points than $\bar{k}$ can be created via a fold bifurcation for appropriate parameter values i.e $\rho = \rho_2$: a stable cycle-2 arises able to attract any orbit originating from an i. c. $k_0 \neq 0$, (see figure 3 panel (a)).

Consider also the set of bifurcations as $\rho \in (-70, -10)$ for suitable values of the other parameters (see figure 3 panel (b)). The fixed point $\bar{k}$ is stable when $\rho > \rho_2$; at $\rho = \rho_2$ it loses stability via flip bifurcation. This bifurcation is the first one of the well known period doubling route to chaos. The properties of the limiting map $g_{\bar{n}}$ are such that when ρ decreases, dynamics may get

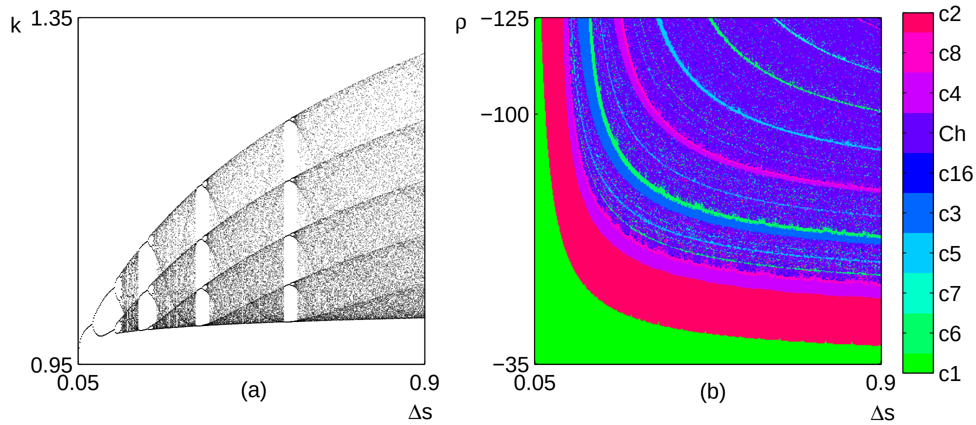

Figure 4: (a) Bifurcation diagrams of map $g_{\bar{n}}$ with respect to Δs at $\delta = 0.05$, $\bar{n} = 0.05$, $s_w = 0.05$ and $\rho = -100$. Complexity increases as Δs increases. (b) Cycle cartogram in $(\Delta s, \rho)$ plane for $\delta = 0.05$, $s_w = 0.05$ and $\bar{n} = 0.05$. Color c-i represents the cycle of period i owned by the map for a given initial condition while color Ch indicates that the trajectory is probably chaotic.

increasingly complicated. At the limit, if $\rho \to -\infty$, map $g_{\bar{n}}$ is approximated by g^l that is topologically chaotic as stated in theorem 4.2.

The arguments used to prove Theorem 4.2 aim at confirming that the quantity $\Delta s = s_r - s_w$ plays an important role with respect to chaotic patterns to be exhibited by the model. First consider that the hypothesis of theorem 4.2 holds if

$$0 < \delta + \bar{n} - s_w < \Delta s \tag{13}$$

and consequently when assuming $\delta + \bar{n} - s_w$ being constant (and, obviously, lesser than one) then as Δs increases relation (13) is going to hold and chaos is likely to emerge as illustrated in figure 4 panel (a) where the bifurcation diagram with respect to Δs is presented for given values of the other parameters.

Furthermore observe that condition (13) is surely verified if Δs is great enough (that is the difference in savings between shareholders and workers is relevant) and that in such a case if ρ is small enough (that is the elasticity of substitution between the two production factors is close to zero) map $g_{\bar{n}}$ is topologically chaotic. Consequently numerical simulations when varying

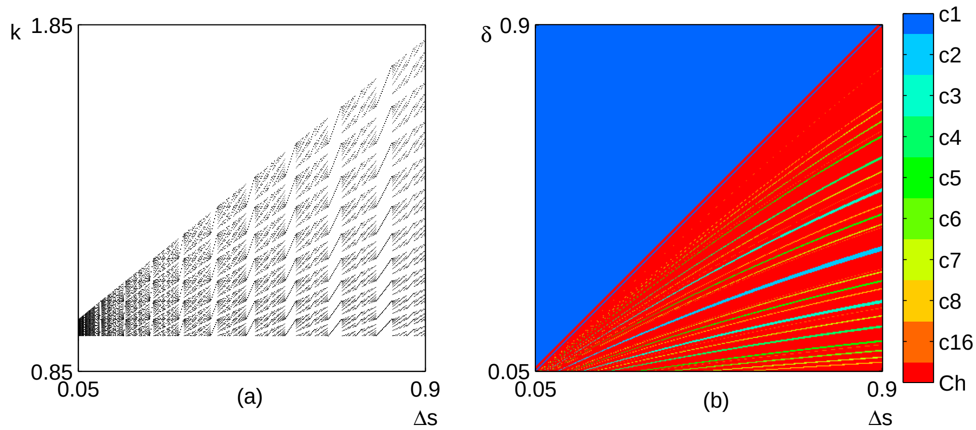

Figure 5: (a) Bifurcation diagrams of map g^l with respect to Δs at $\delta = 0.05$, $\bar{n} = 0.05$, $s_w = 0.05$. (b) Cycle cartogram in $(\Delta s, \delta)$ plane for $\bar{n} = 0.05$ and $s_w = 0.05$. Color c-i represents the cycle of period i owned by the map for a given initial condition while color Ch indicates that the trajectory is probably chaotic.

both Δs and ρ are interesting to be observed.

Figure 4 panel (b) contains a cycle cartogram showing a two parametric bifurcation diagram qualitatively. Each color represents a long-run dynamic behavior for a given point on the parameter plane $(\Delta s, \rho)$ and for a generic initial condition. Also note that, as is typical in one-dimensional bimodal dynamic maps, several period doubling and period halving cascades exist (see Hommes, 1994).

Finally in figure 5 the limiting map g^l as defined in (12) is considered. In such a case $\rho \to -\infty$ so that, given the equilibrium population growth rate $\bar{n}$ and assigned a value (sufficiently small) to the working propensity to save s_w, only parameters δ and Δs have to be considered. In panel (a) the bifurcation diagram with respect to Δs is presented. The numerical simulation confirms that quite complicated dynamics emerges with Leontief technology. In panel (b) both Δs and δ are varying. Obviously only with $\Delta s >> \delta$ chaotic patterns are presented as stated in theorem 4.2.

5 Conclusions

The results of our analysis show that the one-sector growth model with differential savings and non-constant population growth rate, can exhibit fluctuations or even chaotic patterns.

The labor force growth rate is endogenous and described by a generic iterative scheme having a unique positive globally stable equilibrium $\bar{n}$. The study herewith presented aims at confirming the existence of a compact global attractor for system T along the invariant line $n = \bar{n}$. Consequently asymptotic dynamics of growth models with constant population growth rate can be related to those with non-constant population growth if the steady state rate is globally stable.

Furthermore, the role of the production function's elasticity of substitution and its relation to the capital income monotonicity property, have been related to the creation and propagation of complicated dynamics. In fact as the CES elasticity of substitution parameter falls below one, capital income is not surely monotonic and fluctuations may arise. The fixed coefficient production function case is just the extreme example of this observation.

References

[1] Accinelli, E., Brida, J.G., 2005. Re-formulation of the Solow economic growth model with the Richards population growth law. EconWPA at WUSTL 0508006.

[2] Becker, R. A., 2006. Equilibrium Dynamics with Many Agents, in Handbook of optimal growth theory (Dana R.A., Le Van C., Mitra T. and Nihimura K., eds.) Springer-Verlag.

[3] Becker, R. A., Foias, C., 1994. The local bifurcation of Ramsey equilibrium. Economic Theory 4, 719-744.

[4] Beverton, R.J.H., Holt, S.J., 1957. On the dynamics of exploited fish populations. Fishery Investigations 19, 1-533.

[5] Bohm, V., Kaas, L., 2000. Differential savings, factor shares, and endogenous growth cycles. Journal of Economic Dynamics and Control 24, 965-980.

[6] Brianzoni, S., Mammana, C., Michetti, E., 2007. Complex dynamics in the neoclassical growth model with differential savings and non-constant labor force growth. To appear in Studies in Nonlinear Dynamics and Econometrics, 11.1.

[7] Cheban, D., 2004. Global attractors of Non-Autonomous dissipative dynamical systems. World Scientific.

[8] Chiang, A.C., 1973. A simple generalisation of the Kaldor–Pasinetti theory of profit rate and income distribution. Economica 40, 311-313.

[9] Cushing, J.M., Henson, S.M., 2001. Global dynamics of some periodically forced, monotone difference equations. Journal of Difference Equations and Applications 7, 859-872.

[10] Cushing, J.M., Henson, S.M., 2002. A periodically forced Beverton-Holt equation. Journal of Difference Equations and Applications 8(12), 119-1120.

[11] Day, R. H., 1994. Complex Economic Dynamics. The MIT Press, Cambridge, Massachusetts.

[12] Dechert, W. D., 1984. Does optimal growth preclude chaos? A theorem on monotonicity. Zeitchrift fur Nationalokomic 44, 57-61.

[13] Faria, J. R., 2004. Economic growth with a realistic population growth rate. Political Economy Working Paper 05/04.

[14] Kaldor, N., 1956. Alternative theories of distribution. Review of Economic Studies 23, 83-100.

[15] Kaldor, N., 1957. A model of economic growth. Economic Journal 67, 591-624.

[16] Harrod, R. F., 1939. An essay in dynamic theory. Economic Journal 49, 14-33.

[17] Hommes, C., 1994. Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand. Journal of Economic Behavior and Organization, 24, 315-335.

[18] Li, T.Y.,Yorke, J. A., 1975. Period three implies chaos. American Mathematical Monthly 82, 895-992.

[19] Mira, C., Gardini, L., Barugola, A., Cathala, J. C., 1996. Chaotic dynamics in two-dimensional noninvertible maps. World Scientific, Singapore.

[20] Pasinetti, L.L., 1962. Rate of profit and income distribution in relation to the rate of economic growth. Review of Economic Studies 29, 267-279.

[21] Ramsey, F.P., 1928. A mathematical theory of saving. Economic Journal 38, 543-559.

[22] Samuelson, P. A., Modigliani, F., 1966. The Pasinetti paradox in neoclassical and more general models. Review of Economic Studies 33, 269-301.

[23] Schtickzelle, M., Verhulst, P.F., 1981. La premiere decouverte de la function logistique. Population 3, 541-556.

[24] Solow, R.M., 1956. A contribution to the theory of economic growth. Quarterly Journal of Economics, 70, 65-94.

[25] Stiglitz, J.E., 1969. Distribution of income and wealth among individuals. Econometrica, 37, 382-397.

[26] Swan, T.W., 1956. Economic growth and capital accumulation. Economic Record 32, 334-361.