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A stochastic model for financiers

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Temi di discussione n. 42

Maggio 2007

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Abstract

In this work, two models for legal and illegal financiers are presented. The aim of the financiers are different: a bank try to minimize the default probability of the funded company, while the illegal financier aims to bring the company to bankruptcy and, at the same time, to obtain the maximum level of the firm's guarantee wealth. A couple of stochastic dynamics optimization problems are solved. The illegal case let intervene a numerical analysis of the microeconomic situation of the firm, starting from real data and writing new simulation procedure in Matlab and GAMS. The legal case has been solved in closed-form, by using stochastic control theory.

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Introduction

The aim of this paper is to determine the level of the interest rate that allow to maximize the target function of both the legal and the illegal function. We propose two models. The first of these determines the interest rate that maximize the probability of redeeming the loan when the contract is made whit a bank. The second one maximize the probability of the borrower's default when he/she has recourse to the usury market [6]. Italian law defines granting of loans at an interest rate higher than the last one. Such a definition has little economical content. As a matter of fact, the rate at which the loan is released may not be higher than the legal one, but it may even be more convenient, just because often, behind the usury credit - an illegal activity by its very nature - there is hidden a further illegal activity, namely the financial money laundering. According to a definition based on the literature, money laundering may be the described as an economical activity that has the function of transforming a potential purchasing flow, which is the result of illegal activities, and, as such, not directly usable as choices of consume and of investment, into an effective purchasing power. In fact a subject who has at his/her disposal an income from illegal activities cannot use it without facing himself/herself the problem of avoiding to increase the probability of being predicted because of his/her illegal activities. For this reason the above subject has to decide whether and in which percentage of their illegal money it is suitable to cleaning. Among the most used cleaning techniques there is the granting of loans at no interest, whose usefulness for the criminal is made evident by the analysis of at last two factors: on the one hand, it allows him/her to avoid having to do with the Institutions, in particular the banking system, then reducing the probability of the finding out of both the original crime and the illegal activity; on the other hand, the

granting of a loan at no interest allows the criminal to pursue his/her final goal, namely to enter into possession of what has been offered by the borrower - the collateral is the fundamental tool in view of money laundering, and as such, is endowed with a greater intrinsic value - when the borrower is not able to pay back a loan. Indeed, what happens in the case of the borrower's default? In loan granted by a bank the borrower is forced to go bankrupt, thus losing the pledge offered as a guaranty, which has a legal economical value lower than that attributed to it by the borrower; this is both because of the monitoring costs of the investment project, which are charged to financier, and because of the legal costs incurred in the recovery of the pledge. The usurer on the contrary offers the possibility of renegotiating the debt. This advantage offered by the usurer respect to the bank, renders the contract initially stipulated by the illegal financier a loan at no interest, a 'trap for fools'. In a second moment the usurer may change the contract conditions in order to minimize the probability of the borrower's default and, thus, to take possession of the pledge, which usually consist of the financed enterprise itself.

In this work, two models for legal and illegal financiers are presented. The aim of the financiers are different: a bank try to minimize the default probability of the funded company, while the illegal financier aims to bring the company to bankruptcy and, at the same time, to obtain the maximum level of the firm's guarantee wealth. A couple of stochastic dynamics optimization problems are solved. The illegal case let intervene a numerical analysis of the microeconomic situation of the firm, starting from real data and writing new simulation procedure in Matlab and GAMS. The legal case has been solved in closed-form, by using stochastic control theory.

The paper is organized as follows. In the next section, the evolution equation

of the company's wealth is described. Sections 3 and 4 are devoted, respectively, to the analysis of the legal and illegal cases. Last section concludes.

1 The evolution equation

We suppose that the wealth of the firm at time t , denoted by $X(t)$, is described by a stochastic differential equation.

Let us consider a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, where the filtration \mathcal{F}_t is constructed as

$$\mathcal{F}_t = \sigma\left(X(s), 0 \leq s \leq t\right) \cup \mathcal{N}, \quad \forall t \geq 0,$$

with

$$\mathcal{N} := \left\{A \in \mathcal{F} \mid P(A) = 0\right\}.$$

The evolution of firm's wealth is

$$\begin{cases} dX(t) = (\mu X(t) - \alpha_t)dt + \sigma X(t)dW(t), \\ X(0) = X_0 \end{cases} \quad (1)$$

where

- $\mu, \sigma \in \mathbf{R}$ are the related to the drift and diffusion term,
- $\alpha_t \in \mathbf{R}$ is the intensity of the debt's payment at time t ;
- X_0 is an integrable random variable in $[0, K]$ with law π_0 and measurable with respect to \mathcal{F}_0 representing the initial value of the firm,
- $W(\cdot)$ is a standard 1-dimensional Brownian Motion that is independent of π_0 .

We assume the existence of an inverse relation between the risk of the fund project and the restitution interest rate.¹

For sake of simplicity, we assume $X_0 =: x \in (0, K)$ deterministic and constant.

Remark 1 *The boundary values 0 and K represent the absorbing barriers for the dynamic of the wealth of the firm. When the value 0 is attained, we are in the situation of the failure of the firm. K represents the case of total restitution of the debt by the firm.*

Remark 2 *There exists a unique solution for the controlled equation (1) (we remind the reader, for example, to Øksendal (1995)).*

By Remark 2, and fixed $x \in (0, K)$ and $\alpha \in \mathbf{R}$, we denote the unique solution of (1) as $X_x^\alpha(\cdot)$.

Moreover, let us denote with \mathcal{T} the set of the stopping times in $[0, +\infty)$, i.e.

$$\mathcal{T} := \{\tau : \Omega \rightarrow [0, +\infty) \mid \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}, \quad (2)$$

and let us define the exit time $\tau_{(0,K)}$ of the dynamic from $(0, K)$ as

$$\tau_{(0,K)} := \inf \left\{ t \geq 0 \mid X_x^\alpha(t) \notin (0, K) \right\}, \quad (3)$$

Remark 3 *Standard results provide the measurability of $\tau_{(0,K)}$ with respect to the σ -algebra \mathcal{F} . Indeed, it results $\tau_{(0,K)} \in \mathcal{T}$.*

¹We remind to [13], extending to the risk-neutral agents the Assumption of Stiglitz and Weiss on the risk-adverse behavior of the funded firms with respect to the interest rates applied to the financiers.

2 The optimization problem [legal case]

In the legal case the guarantees are irrelevant. In fact, in presence of bankruptcy, the financier loses completely the credit grant. Hence, we assume in this context that $g(t) \equiv 0$, for each t .

A company is funded by a bank only under obvious profit conditions.

So we write the value function V as

$$V(x) := \min_{\alpha \in \mathcal{A}} P(X_x^\alpha(\tau_{(0,K)}) = 0), \quad (4)$$

where the admissible region of the problem is

$$\mathcal{A} := \left\{ \alpha : [0, +\infty) \times \Omega \rightarrow [\delta_1, \delta_2] \subset \mathbf{R}^+ \text{ such that} \right. \\ \left. K_1 + \left[\int_0^{\tau_{(0,K)}} \alpha_t e^{-\delta t} dt \right] \geq K_2, \alpha_t \in \mathcal{F}_t, \forall t \geq 0 \right\}. \quad (5)$$

K_1 and K_2 are nonnegative constants: K_1 is the amount of the original debt of the firm while K_2 is the threshold representing the minimal profit that the financier aims to obtain. δ is the discounted factor, while δ_1 and δ_2 are, respectively, the lower and the upper bounds for the interest rate.

2.1 Hamilton-Jacobi-Bellman Equation

The following result holds.

Theorem 4 (HJB Equation) *Suppose that $V \in C^2((0, K)) \cap C^0([0, K])$.*

Then

$$\sup_{a \in [\delta_3, \delta_4]} \left\{ (\mu x - a)V'(x) + \frac{1}{2} \sigma^2 x^2 V''(x) \right\} =: \sup_{a \in [\delta_3, \delta_4]} H_a(x) = 0, \forall x \in (0, K), \quad (6)$$

with the boundary conditions

$$V(0) = 0, \quad V(K) = H, \quad (7)$$

where H is a positive constant relative to the income of the financier, when the debt is completely paid.

In the next result we analyze the problem of existence, uniqueness and closed form for the solution of the Hamilton Jacobi Bellman Equation.

Theorem 5 *The following conditions hold.*

1. *There exists $a^* \in [\delta_1, \delta_2]$ such that*

$$H_{a^*}(x) = \sup_{a \in [\delta_1, \delta_2]} H_a(x),$$

for each $x \in [0, K]$.

2. *There exists an unique solution of the equation (6)-(7).*

3. *The function*

$$V(x) = C \int_0^x \exp\left\{-\frac{2a^*}{\sigma^2 t} - \frac{2\mu}{\sigma^2} \log t\right\} dt, \quad (8)$$

with

$$C = H \cdot \left(\int_0^K \exp\left\{-\frac{2a^*}{\sigma^2 t} - \frac{2\mu}{\sigma^2} \log t\right\} dt \right)^{-1},$$

is the solution of the Hamilton Jacobi Bellman Equation (6)-(7).

Proof.

1. Fixed $x \in [0, K]$, then $H_a(x)$ is a continuous function of the variable a in the compact set $[\delta_1, \delta_2]$. By Weierstrass' Theorem, we obtain the thesis.
2. The proof comes out from the standard existence and uniqueness theorem for systems of first order ordinary differential equations.

3. Let us consider a^* the same as in the first point of this theorem. (6)

can be written as

$$\begin{cases} (\mu x - a^*)\gamma(x) + \frac{1}{2}\sigma^2 x^2 \gamma'(x) = 0, & x \in (0, K), \\ \gamma(x) = V'(x), & x \in (0, K). \end{cases} \quad (9)$$

By separating the variables, the first order differential equation in (9) becomes

$$\frac{d\gamma}{\gamma} = \frac{2(a^* - \mu x)}{\sigma^2 x^2} dx,$$

and then

$$\log \gamma(x) = -\frac{2a^*}{\sigma^2 x} - \frac{2\mu \log x}{\sigma^2} + C_1, \quad C_1 \in \mathbb{R}.$$

Therefore we obtain

$$\gamma(x) = C_2 x^{-\frac{2\mu}{\sigma^2}} e^{-\frac{2a^*}{\sigma^2 x}} \quad C_2 \in \mathbb{R}^+.$$

These computations provide

$$V(x) = \int_0^x \gamma(t) dt = C_2 \int_0^x t^{-\frac{2\mu}{\sigma^2}} e^{-\frac{2a^*}{\sigma^2 t}} dt + C_3, \quad C_3 \in \mathbb{R}.$$

By imposing the boundary conditions (7), we get the thesis.

■

Corollary 6 *The value function V is strictly increasing in $(0, K)$.*

Proof. By the previous theorem, we have that V is twice differentiable in $(0, K)$. Moreover, it results:

$$V'(x) = C x^{-\frac{2\mu}{\sigma^2}} e^{-\frac{2a^*}{\sigma^2 x}} > 0,$$

being

$$C = H \cdot \left(\int_0^K \exp\left\{ -\frac{2a^*}{\sigma^2 t} - \frac{2\mu}{\sigma^2} \log t \right\} dt \right)^{-1} > 0.$$

That is what we want to prove. ■

2.2 Optimal strategies

Our starting point, for this section, is the Hamilton-Jacobi-Bellman Equation stated in the previous sections.

With the Verification Theorem, we want to formalize the presence of optimal strategies for this particular control problem. We enunciate it. For the proof, we remind the reader to Fleming and Soner, 1993.

Theorem 7 (Verification Theorem) *Assume that $u \in C^0([0, K]) \cap C^2((0, K))$ be a solution of (6)-(7).*

Then it results

- (a) $u(x) \geq V(x), \forall x \in [0, K]$.
- (b) *Let us consider (α^*, x^*) an admissible couple at x such that*

$$\alpha^* \in \operatorname{argmax}_a \left\{ (\mu x^*(t) - a)V'(x^*(t)) + \frac{1}{2}\sigma^2(x^*(t))^2 V''(x^*(t)) \right\}.$$

Then $(\alpha^, x^*(t))$ is optimal at x and it results $u(x) = V(x), \forall x \in [0, K]$*

Now we can describe the optimal strategies associated to this control problem. The optimal controls α^* are bang-bang controls. We get, by the analysis of HJB Equation,

$$\alpha^*(x) = \begin{cases} \delta_2 & \text{for } x \mid V'(x) < 0 \\ \delta_1 & \text{for } x \mid V'(x) > 0 \\ \text{arbitrary} & \text{for } x \mid V'(x) = 0 \end{cases} \quad (10)$$

Since V is strictly increasing, we have that

$$\Gamma := \{x \in [0, K] \mid V'(x) \leq 0\} = \emptyset.$$

Therefore,

$$\alpha^*(x) = \delta_1,$$

for each $x \in [0, K]$.

3 The optimization problem [Illegal case]

A key role in the usury model is played by the guarantee g . The guarantee is related to the wealth of the firm and takes into account the eventual income obtained by the financier in company's bankruptcy case. Therefore it seems to be obvious to assume the guarantee nonnegative. We introduce it. We define

$$g : [0, +\infty) \times \Omega \rightarrow [0, G],$$

through a stochastic differential equation as follows:

$$\begin{cases} dg(t) = \gamma_1(g(t), X(t), \alpha_t)dt, \\ g(0) = g_0 \end{cases} \quad (11)$$

where

- γ_1 is a real function,
- $g_0 > 0$ is deterministic.

g_t is the subjective wealth of the firm evaluated at time t by the financier. It is assumed to evolve deterministically. γ_1 in (11) is increasing with respect to $X(t)$ and $g(t)$, and it is decreasing with respect to $\alpha(t)$.

We assume that the nonnegative dynamic g of the guarantee process is bounded by a deterministic threshold G . The objective of the illegal financier

is to maximize the default probability of the company and, contemporaneously, the probability that the guarantee reach the highest level G . Therefore, the value function is

$$V(x) := \max_{\alpha \in \mathcal{A}} [P(X_x^\alpha(\tau_{(0,K)}) = 0) + P(g_{g_0}^\alpha(\tau_{(0,K)}) = G | X_x^\alpha(\tau_{(0,K)}) = 0)], \quad (12)$$

where the admissible region of the problem is

$$\mathcal{A} := \left\{ \alpha : [0, +\infty) \times \Omega \rightarrow [\delta_3, \delta_4]; \text{ such that } \alpha_t \in \mathcal{F}_t, \forall t \geq 0 \right\}. \quad (13)$$

Although it is generally the upper bound of the usury interest rate infinity, empirical evidence shows that $\delta_4 = 500\%$ with only the 9% of the event that overcome such high threshold². The lower bound is assumed to be $\delta_3 = 0$, in agreement with the purposes of the illegal financier (to construct a trap for the company, to reinvest illegal money). We have built the α range of variation on such values and we have added this further constrain to the forthcoming maximization model, that has been implemented using GAMS.

Our goal is to derive the average interest rate α that can maximize the objective function V .

The first step is to generate randomly a vector α of interest rates from a Poisson distribution with parameter $\lambda = 4$,³ in agreement with the previous literature.

Then, with the same language, for all generated value α_i we derive by equation (1), actually with a Monte Carlo simulation, the trajectory of $X_t^{\alpha_i}$. At

²CENTRO STUDI E RICERCHE SULLA LEGALITA' E CRIMINALITA' ECONOMICA, L'usura tra vecchi confini e nuovi mercati, Roma - 2002.

³To this scope, we have used the MatLab generator; the vector have $n = 1000$ components.

this aim we fix the values $\mu =$, σ , X_0 referring to real data⁴. We discretize as usual the Brownian Motion $dW(t) = \Lambda * \sqrt{\Delta t}$, where Λ is a random number extracted by a centered normal distribution.

For each $X_t^{\alpha_i}$ we derive the time τ^{α_i} in which, for the first time, the trajectory of $X_t^{\alpha_i}$ hits the barrier $\{0, K\}$. The threshold K is defined by the existing empirical relation $K = \beta X_0$ ⁵.

Then, we calculate $\tau^*(0, K) = \frac{1}{n} \sum_{i=1}^n \tau^{\alpha_i}$. For the illegal financier it is reasonable to suppose that there is an inverse relation between G , such as how much he estimates the wealth of the firm at the beginning of the contract, and $X(0)$ ([6, 3]); due to this reason, as $G \rightarrow \infty$, then $X(0) \rightarrow 0$. On the other side, it is well known that $X(0) = \frac{X(t)}{(1+\alpha)^t}$, so it is easy to obtain the objective function that we have to maximize to realize the illegal financier goal (12).

We approximate $V(X)$ of the illegal financier as

$$V(x) = \max_{\alpha \in \mathcal{A}} \frac{(1 + \alpha)^t}{X(t)}.$$

The set of constraints is

$$\left\{ \begin{array}{l} G = \frac{(1 + \alpha_t)^t}{X(t)}, \\ X(t) = X(t-1) + \{[\mu X(t-1) - \alpha_t] \Delta t + \sigma X(t-1) \cdot \Lambda \sqrt{\Delta t}\}. \end{array} \right.$$

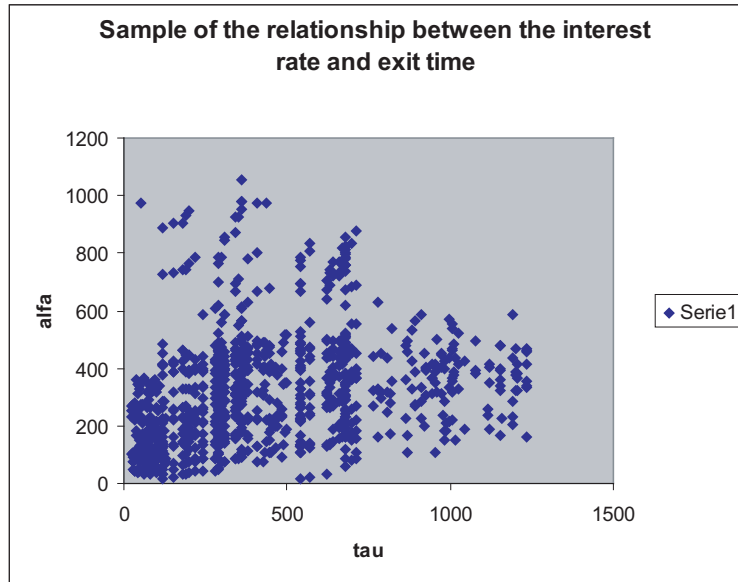
We consider $\tau^*(0, K)$ to define the Δt value for each time.

As a result we have obtained that the average level α of the interest rate need to obtain the purpose of the illegal financier coincides with the upper

⁴In particular, we consider $\mu = 1 + \rho = 1.001$, where ρ is the revaluation rate of the company, $\sigma = 0.01$ and $X_0 = 950000$.

⁵ β comes out from the the ratio between G and X_0 . The empirical mean value of X_0 is 950000. Therefore, $G = 1325000$ and $\beta = 1.42$. The data are available under request. The Source does not want to be nominated in the present paper.

bound of its variation range; in particular we have found that the level of α increases while Δt decreases (see Figure).



4 Conclusions

In this work the best interest rate, that a financier has to apply to a funded company to maximize an objective function, has been explored. The distinguishing between legal and illegal financier has been proposed. At this aim we construct a couple of models. The scope of the legal financier is to maximize the probability that the funded firm will restitute totally its debt, while the illegal financier try to maximize at the same time the default probability of the company and the guarantee related to company's wealth. The solution strategies of the models are different: in the legal case, an analytical closed form solution is showed; the illegal problem has been solved by using a numerical approach. The numerical approach is due to a consistent and not

easy formalization of the usury-type situation, that does not allow a better tractability.

The best interest rate is the maximum one in the illegal case and the minimum one in the legal financier model. In the former model the interest rate is as close as possible to the maximum one as well as an eventual renegotiation date approaches the mean τ^* .

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