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Abstract

This paper presents a new stochastic model of asset pricing, based on agents with heterogeneous beliefs. Forecasting rules of all agents are characterized by a stochastic term that works as an agent-based time dependent weight of the conditional expectation of the fundamental. Since we consider the presence of an imitative behavior between agents, these weights depend stochastically on the type-distribution of agents. The resulting dynamical system is firstly analyzed in a deterministic framework. Starting from the results obtained in the deterministic case, the model is lastly explored by reintroducing randomness. The deterministic study aims at providing the existence of a region in the parameters plane where the unique possible dynamics is the convergence to a steady state, while complexity is exhibited outside such region. This region is also analyzed by reintroducing stochasticity and we provide an explicit formula for its probability measure. Our findings are in agreement with the economic meaning of the parameters. Finally, we propose a bayesian analysis, in order to explore the distribution of the adjustment term of the proportion of agents.

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1 Introduction

The traditional approach in economics and finance is based on a representative agent who is assumed to be *rational*, where rational behavior covers two different aspects. Firstly, a rational agent makes an optimal decision that maximizes utility or profit, taking into account the choices of the other economic actors. Secondly, the agent has rational expectations (RE) about future variables, in other words, the agent's prediction of future variables coincides with the mathematical expectations conditional on the information set available. These requirements imply that a rational decision maker knows the market equilibrium equations and is able to solve the model. Yet, as has been argued by Simon (1957), knowledge about the economic environment and the model equations is an unrealistic assumption because it is too restrictive. Moreover in most nonlinear market equilibrium models it is hard to compute the rational expectations equilibrium, even supposing the agent knew all the equilibrium equations.

In recent years, several models in which markets are populated by boundedly rational, heterogeneous agents have been proposed as an alternative to rational expectations (see Conlisk (1996) for a more detailed explanation of bounded rationality). This change involves the following three important aspects. First of all the shift from a representative agent to heterogeneous agent systems. Kirman (2006) argues that heterogeneity plays an important role in economic models and summarizes some of the reasons why the assumption of heterogeneous agents should be considered. One argument is that in a market with homogeneous agents there will be "no trade" because trade will not take place between identical individuals. Another argument considers an evolutionary explanation according to which adaptation and evolution necessarily involve heterogeneity. Second, it is obvious that heterogeneity implies a shift from simple analytically tractable models with a representative, rational agent to a more complicated framework and then a computational approach becomes necessary. Finally, as underlined e.g. in Hommes (2001), in a heterogeneous world, full rationality is impossible since it requires perfect knowledge of the beliefs of other agents and then *bounded rationality* takes place. The most important feature of this approach is that bounded rationality does not require capability to solve the model and it implies that beliefs and realizations of future variables do not necessarily coincide. More precisely, a boundedly rational agent is modeled as able to form expectations based on an observable information set and he adapts his predictions when some other information becomes available. In order to take decisions, the boundedly rational agent uses simple rules of thumb and adapts his choices when he learns about the economic environment. The resulting *Adaptive*

Learning may or may not converge to a rational expectations equilibrium.

An interesting contribution in the literature evaluating these arguments is given by the theory of evolving selection of predictions, or learning strategies, due to Brock and Hommes (1997) and (1998). The authors propose simple, analytically tractable heterogeneous agent models to show that irrational strategies can survive evolutionary selection. They studied heterogeneity in expectation formation by introducing the concept of *Adaptive Belief System* (ABS) to model economic and financial models. More specifically, different types of agents have different beliefs about future variables and prediction selection is based upon a performance measure which is available to all agents. Furthermore, the fraction of agents employing each rule is assumed evolving over time instead of being fixed.

In Brock and Hommes (1998) the authors investigate adaptive beliefs in the present discounted value asset pricing model and find complicated dynamics for a large intensity of choice, with an irregular switching between close to fundamental fluctuations and upward or downward price trends. As in many financial models with heterogeneous agents, Brock and Hommes (1998) assume two typical investor types, i.e. *fundamentalists* and *chartists*. Fundamentalists believe that the price of an asset is determined by its fundamental value. They sell (buy) assets when their prices are above (below) the fundamental value. In contrast, chartists or technical analysts do not take the fundamental value into account but look for trends in past prices, and prediction is based upon simple trading rules (see Hommes (2006) for an extensive survey of heterogeneous agent models).

In this paper heterogeneity is assumed, in fact agents choose different predictors of future values of endogenous variables. In particular, the model we present aims to take into account two different aspects of this heterogeneity. First of all, beliefs of all agents are affected by a deterministic additive error, depending on the known previous deviation between theoretical and observed fundamental. This approach is in agreement with the heterogeneous beliefs model in Brock and Hommes (1998). Second, and differently with respect to Brock and Hommes (1998), we introduce the existence of a stochastic term which works as an agent-based time dependent weight of the conditional expectation of the fundamental. By assuming that such a weight is unitary, independently of time and agent type, we find the model of Brock and Hommes (1998).

Furthermore, the model we present takes into account a commonly accepted feature of financial markets: the presence of an imitative behavior among agents. Investors can decide to herd, instead of acting according to the information obtained from market analysis. This phenomenon provides an explanation for the reason for the misalignment between prices and asset values. A recent research field has been developed in order to explain herd behavior among rational agents, through several papers. In Banerjee (1992), Bickhchandani et al. (1992), Chamley and Gale (1996) and Chari and Kehoe (2000), the role of knowledge in markets has been used to study the learning effects of the sequential actions of agents. Avery and Zemsky (1998) proved that it is better to believe in private information than imitate other agents. In our financial market, we model the imitative behavior by assuming that weights depend stochastically on the type-distribution of agents.

In order to develop the general model, we rely on a market populated by fundamentalist and chartist agents. The stochastic dynamical system we obtain is firstly bi-dimensional, but it can be reduced to a one-dimensional one. The state variable is the deviation between theoretical and observed fundamental, and the evolution of the map is studied with respect to a couple of relevant parameters: the conditional expectation of the fundamental (δ) and the adjustment term of the proportion of agents (β). The parameter δ represents what the market expects to gain in the next period from the fundamental, while β provides information on the movements among fundamentalist and chartist groups, based on the previous realization of the fundamental. The joint analysis of the map with respect to δ and β is important in understanding the relationship between the proportion of fundamentalists and forecasts on the fundamental value. In particular, complex dynamics is attained for some values of δ and β. More precisely, if δ and β are both small (or large) enough, there exists a *stability region* in the parameters' plane (δ, β) for which the unique possible dynamics is the convergence to a steady state. The complementary of the stability region is characterized by a large variety of cycles of different orders. Notice that if δ and β are both small (large), then most agents are chartists (fundamentalists) in type and the forecast on the fundamental provides small (high) values. We will prove that in such cases only simple dynamics is possible, while the system shows complex dynamics in the remaining cases.

After the deterministic study, the stability region is also analyzed by reintroducing randomness. In particular, we provide an explicit formula for its probability measure, starting from a numerical analysis of its boundaries. Then, we propose a bayesian analysis, in order to explore the distribution of β and suggest two further research lines: the first one concerns the discovery of an eventual presence of change in persistence of the time-dependent stochastic process describing the adjustment term of the proportion of agents; the second one is related to a possible performing of cluster analyses of the parameter β.

The remaining part of this paper is organized as follows. In the next section, both the general model and the fundamentalist-chartist model are presented. Section 3 is devoted to the analysis of the deterministic skeleton of the dynamical system. In section 4 some stochastic features of the model are described, in agreement with the deterministic analysis. The final section concludes.

2 The asset pricing model

The starting point of our model is the framework proposed by Brock and Hommes (1998). The asset pricing model is composed of one risky asset, whose price (ex dividend) per share at time t is p_t, and one risk free asset which is perfectly elastically supplied at gross return $R = 1 + r_f$. Let $\{y_t\}$ be the stochastic dividend process of the risky asset. The wealth's dynamics W is described by:

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t,$$

where z_t denotes the number of shares of the asset purchased at date t.

Let us denote with E_t, V_t the conditional expectation and conditional variance operators based on a publicly available information set consisting of past prices and dividends. Heterogeneity is introduced via the assumption that agents have different beliefs about the future price of the risky asset. Then, let E_{ht}, V_{ht} be the beliefs of investor type h about the conditional expectation and conditional variance.

The excess return is $p_{t+1}+y_{t+1}-Rp_t$, and we assume that beliefs about the conditional variance of excess returns are constant and the same for everyone, i.e.

$$V_{ht}(p_{t+1} + y_{t+1} - Rp_t) = \sigma^2$$

for all types h. Assume that each investor type is a myopic mean variance maximizer, then for type h the demand for shares z_{ht} solves

$$\max_z \left\{ E_{ht}(W_{t+1}) - \frac{a}{2} V_{ht}(W_{t+1}) \right\},$$

i.e.

$$z_{ht} = \frac{E_{ht}(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2}$$

where a denotes the risk aversion, which is assumed to be equal for all traders. Let s_t denote the supply of shares and n_{ht} the fraction of investors of type h at time t. Equilibrium between demand and supply implies

$$\sum_h n_{ht} \left\{ \frac{E_{ht}(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2} \right\} = s_t. \qquad (1)$$

If there is only one investor type, market equilibrium yields the pricing equation

$$Rp_t = E_{ht}(p_{t+1} + y_{t+1}) - a\sigma^2 s_t. \tag{2}$$

Now consider equation (2) in the special case of zero supply of outside shares, i.e. $s_t = 0$, for all t.[1] In other words we analyze a secondary market where we do not have issue of new shares. Furthermore, in order to get a benchmark notion of the rational expectations fundamental solution p_t^*, consider the information set $\mathcal{F}_t$ and the equation

$$Rp_t^* = E_t(p_{t+1}^* + y_{t+1}), \tag{3}$$

where E_t is the conditional expectation on the information set $\mathcal{F}_t$.

We define the deviation x_t from the benchmark fundamental p_t^* as

$$x_t = p_t - p_t^*. \tag{4}$$

Rewrite (1) for the case of zero supply of outside shares to get

$$Rp_t = \sum_h n_{ht} E_{ht}(p_{t+1} + y_{t+1}). \tag{5}$$

Let us introduce the new ingredient in the class of beliefs about deviations from the fundamental solution. We make the following assumption:

Assumption 2.1. *All beliefs are of the form*

$$E_{ht}(p_{t+1} + y_{t+1}) = g_{ht} \cdot E_t(p_{t+1}^* + y_{t+1}) + f_{ht}(x_{t-1}), \tag{6}$$

where p_t^* denotes the fundamental, $E_t(p_{t+1}^* + y_{t+1})$ is the conditional expectation of the fundamental on the information set $\mathcal{F}_t$, $x_t = p_t^* - p_t$ is the deviation from the fundamental, f_{ht} is a deterministic function and g_{ht} is a stochastic process which can differ across trader types h.

Differently from Brock and Hommes (1998) in equation (6) we introduce the term g_{ht}, that represents a stochastic adjustment factor of the conditional expectation of the fundamental, given the information available at time t. Due to the presence of an imitative behavior between agents, we can suppose that g_{ht} depends randomly on the proportion of agents of type h at time t. We can assume:

$$g_{ht} = \alpha_h n_{ht} + \gamma_{ht}, \tag{7}$$

for any investor type h, where $\alpha_h \in \mathbb{R}$ and γ_{ht} are i.i.d. stochastic processes such that γ_{ht} is independent on γ_{kt}, for any investor type $h \neq k$.

[1] In making this assumption we follow Brock and Hommes (1998).

2.1 Fundamentalist and chartist agents

This subsection aims at providing an analysis of a market populated only by two types of agent, whose beliefs are driven by a fundamentalist and a chartist viewpoint. This subcase of the general model described above is not restrictive, since an agent, taking a position in a market, makes forecasts about the future price of an asset basically by performing a technical analysis of the market (chartist approach) or by analyzing the value of the market's fundamental (fundamentalist approach).

Let us denote with f and c the indexes related to the fundamentalist and the chartist groups. The proportions of agents, n_{ft} and n_{ct}, vary during the time. This explains the empirical fact that the interaction between agents implies the presence of an imitative behavior, which is a commonly accepted feature in financial markets and it represents the focus of a wide part of financial research. Therefore we have switch phenomenon of the agent forecasts' approach, that implies a change in the adjustment factors g's as time varies. The size of such inversions provides information on the price dynamics. In fact, agents are not purely price takers, but they can modify the path of the price evolution of a risky asset.

Since the g's represent an adjustment of the conditional expectation of the benchmark fundamental, we can provide an explicit shape of these terms in the fundamentalist (g_f's) and chartist (g_c's) cases.

The adjustment factor in the fundamentalist case has to reflect two conditions:

- the growing confidence in the expectation of the fundamental, as the number of fundamentalists increases;
- the overestimate of the expectation of the fundamental. This fact implies that g_{ft} is basically greater than 1.

As a consequence, g_{ft} can be formalized as follows:

$$g_{ft} = kn_{ft} + \gamma_t, \tag{8}$$

where $k \in (0, 1)$ and γ_t i.i.d. such that $E[\gamma_t] = 1$.

On the contrary, the chartist adjustment factor is negative, with increasing absolute value as the number of chartists grows. The countertendency with respect to the fundamentalist point of view allows us to use the same proportionality parameter k, and to define g_{ct} as

$$g_{ct} = -kn_{ct}. \tag{9}$$

Let us define the difference between the adjustment factors at time t by introducing m_t as follows:

$$m_t = g_{ft} - g_{ct}, \tag{10}$$

where g_{ft} and g_{ct} are defined in (8) and (9).

Moreover, taking into account the presence of an imitative behavior among agents in the market, we assume that the proportion of agents of each group depends on the known deviation from the fundamental and on the size of the switch from the fundamentalists' adjustment factor to the chartists' one at the previous time. These assumptions are totally in line with the presence of an imitative behavior among agents in the market. The dependence on the term x_{t-1} is in agreement with the fact that investors try to get information from the errors made at the previous time, that are measured by the deviation from the fundamental at time $t-1$. To keep the model as general as possible, such dependence is assumed to show randomness. We suppose that there a stochastic process β_t exists with support in $(0, +\infty)$ such that

$$n_{ct} = \exp\{-(\beta_t x_{t-1}^2 + m_{t-1}^2)\}, \qquad n_{ft} = 1 - n_{ct}. \tag{11}$$

From Assumption 2.1 and formula (5), we obtain the following equilibrium equation:

$$Rx_t = n_{ct}[(g_{ct}-1)E_t[p^*_{t+1}+y_{t+1}]+f_{ct}(x_{t-1})]+n_{ft}[(g_{ft}-1)E_t[p^*_{t+1}+y_{t+1}]+f_{ft}(x_{t-1})], \tag{12}$$

where g's and n's are defined, respectively, in (7) and (11). If we introduce the random variable $\delta_{t+1} = E_t[p^*_{t+1} + y_{t+1}]$, then (12) becomes

$$Rx_t = n_{ct}[(g_{ct} - 1)\delta_{t+1} + f_{ct}(x_{t-1})] + n_{ft}[(g_{ft} - 1)\delta_{t+1} + f_{ft}(x_{t-1})]. \tag{13}$$

The bivariate stochastic dynamical system we wish to study is obtained by combining equations (10) and (13). After the opportune substitutions, we get:

$$T := \begin{cases} Rx_t = e^{-(\beta_t x_{t-1}^2 + m_{t-1}^2)}\{[-ke^{-(\beta_t x_{t-1}^2 + m_{t-1}^2)} - 1]\delta_{t+1} + f_{ct}(x_{t-1})\} + \\ (1 - e^{-(\beta_t x_{t-1}^2 + m_{t-1}^2)})\{[k(1 - e^{-(\beta_t x_{t-1}^2 + m_{t-1}^2)}) + \gamma_t - 1]\delta_{t+1} + f_{ft}(x_{t-1})\} \\ \\ m_t = k + \gamma_t \end{cases} \tag{14}$$

The following sections contain the analysis of the dynamical system formalized in (14). The study will be conducted in two steps. First of all, in section 3, we analyze the deterministic skeleton of the map, by reproducing the parameters of the system in a deterministic framework. Then, in section 4 we return to the stochastic setting by exploring the randomness of the system, maintaining the analysis in line with the deterministic findings. To reach this goal, the asymptotic distributions of the stochastic parameters are derived in agreement with the stability features of the deterministic equilibria.

3 The deterministic skeleton

In order to focus on the deterministic non-linear asset price dynamics, we can state some hypotheses on the stochastic parameters intervening in the model. Since $E[\gamma_t] = 1$, for each t, we can assume consistently that $\gamma_t = 1\ \forall t$. Moreover, δ_t represents the conditional expectation of the fundamental. From equation (3), it follows that $E_t[\delta_{t+1}] > 0$, and as a consequence we can assume that δ_t is a stochastic process with positive support. In the deterministic framework, we define $\delta_t = \delta > 0$. Finally, we stress that β_t is a stochastic process with support in $(0, +\infty)$, therefore we assume that $\beta_t = \beta > 0\ \forall t$.

Hence, we concentrate the analysis on the case with $f_{ht}(x_{t-1}) = a_h x_{t-1} + b_h$, i.e. the deterministic function of the belief system is linear for each agent-type h. More precisely, we investigate our asset pricing model with two types of beliefs ($h = f, c$) where type 1 are fundamentalists, believing that the price returns to its fundamental value ($a_f = b_f = 0$), whereas type 2 are pure trend chasers ($a_c = 1, b_c = 0$).

Taking into account the previous assumptions on the parameters, system (14) is given by:

$$T := \begin{cases} Rx_t = e^{-(\beta x_{t-1}^2 + m_{t-1}^2)}\{[-k \cdot e^{-(\beta x_{t-1}^2 + m_{t-1}^2)} - 1]\delta + x_{t-1}\} + \\ +(1 - e^{-(\beta x_{t-1}^2 + m_{t-1}^2)})[k(1 - e^{-(\beta x_{t-1}^2 + m_{t-1}^2)}) + \gamma - 1]\delta \\ \\ m_t = k + \gamma \end{cases} . \tag{15}$$

Remark 3.1. *System (15) reduces to the one-dimensional dynamic system given by:*

$$x_t = \phi(x_{t-1}) = \frac{1}{R}\left[e^{-(\beta x_{t-1}^2 + (k+1)^2)}(x_{t-1} - \delta(1 + 2k)) + \delta k\right]. \tag{16}$$

The parameters we want to consider in order to study the dynamic evolution of the map (16) are given by β and δ. This choice is driven by the financial meaning of δ and β. In fact δ is the conditional expectation of the fundamental, representing what the market expects to gain in the next period from the fundamental, given the available information set. On the other hand, β is the adjustment term for the proportions of agents, and it is related to the deviation between theoretical and observed fundamental. Roughly speaking, the parameter β provides information on the movements among fundamentalist and chartist groups, based on the previous realization of the fundamental. More specifically, high values of β involve an increasing proportion of fundamentalists. It is clear from (11) that the special limiting case $\beta \to +\infty$ corresponds to the case where in each period all agents choose the most *sophisticated* predictor, i.e. fundamentalist beliefs. The joint analysis of the map with respect to δ and β explains the relationship between changes in the proportion of fundamentalist agents and forecasts on the fundamental value.

In the following, we state some general results for the map (16). In particular, we first analyze the number of fixed points owned by the map and their stability and, then, we concentrate our attention on several bifurcations which route to complexity.

Lemma 3.2. *(Existence and stability of steady states)*
Let $k \in (0,1)$. Then $\exists k_1 \in (0,1)$ such that $\forall \delta > 0, \beta > 0$:

1. *if $k \geq k_1$ there are two possibilities:*
   - *(a) there is one globally stable steady state $x_1^\star \geq 0$,*
   - *(b) there are three steady states $0 \leq x_1^\star < x_2^\star \leq x_3^\star < \frac{\delta k}{R}$, the steady state $x_2^\star$ is unstable and $x_1^\star, x_3^\star$ are (locally) stable.*

   In the case $k = k_1$, the steady state $x_1^\star$ is the fundamental ($x_1^\star = 0$).

2. *if $k < k_1$ there are two possibilities:*
   - *(a) there is one steady state $x_1^\star < 0$,*
   - *(b) there are three steady states $x_1^\star < 0$ and $0 < x_2^\star \leq x_3^\star < \frac{\delta k}{R}$, the steady state $x_2^\star$ is unstable and $x_3^\star$ is (locally) stable.*

Proof. In order to prove the existence of steady states, consider the following equation:

$$\phi(x^\star) = x^\star \tag{17}$$

where function ϕ is defined as in equation (16). From equation (17) we get that a steady state $x^\star$ must satisfy $e^{\beta x^{\star 2}+(k+1)^2} = \frac{x^\star - \delta(1+2k)}{Rx^\star - \delta k}$. Define $g(x) = e^{\beta x^2+(k+1)^2}$ and $h(x) = \frac{x-\delta(1+2k)}{Rx-\delta k}$ $\forall x < \frac{\delta k}{R}$, then straightforward computations yield the following results:

- $x = 0$ is the unique critical point of g which is a minimum point,

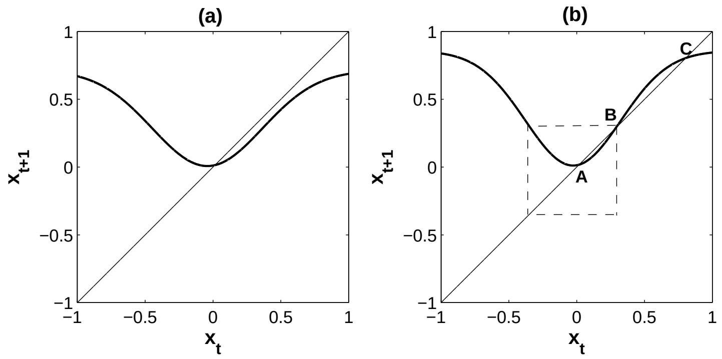

Figure 1: Map (16) for $k = 0.3 \geq k_1$ and $R = 1.05$. (a) For $\beta = 3$ and $\delta = 2.5$ a unique positive globally stable fixed point exists. (b) For $\beta = 4$ and $\delta = 3$ there are three positive fixed points: point B is a repelling fixed point, while A and C are locally stable.

- $h(x)$ is positive and strictly increasing and such that $\lim_{x\to-\infty} h(x) = \frac{1}{R}$ while $\lim_{x\to\frac{\delta k}{R}^-} h(x) = +\infty$.

Hence, if $g(0) \geq h(0)$ both the following cases are possible: there exists a unique steady state (which is the fundamental $x_1^\star = 0$ in the case $g(0) = h(0)$, while $0 < x_1^\star < \frac{\delta k}{R}$ in the case $g(0) > h(0)$) or there are two more steady states $0 < x_2^\star \leq x_3^\star < \frac{\delta k}{R}$. Similarly, if $g(0) < h(0)$ there are two possibilities: one steady state $x_1^\star < 0$ or three steady states $x_1^\star < 0$, $0 < x_2^\star \leq x_3^\star < \frac{\delta k}{R}$. Equation $g(0) = h(0)$ (i.e. $e^{(k+1)^2} = \frac{1+2k}{k}$) has a unique solution k_1, being functions on the left and on the right side increasing and decreasing w. r. t. $k \in (0,1)$, and $\left(e^{(k+1)^2} < \frac{1+2k}{k}\right)_{|k\to 0}$, $\left(e^{(k+1)^2} > \frac{1+2k}{k}\right)_{|k=1}$.

The stability results follow directly from graphical analysis of function $\phi(x)$. More precisely, $\phi(x)$ has one minimum point $x_m < 0$ and one maximum point $x_M > 0$. Moreover, $\lim_{x\to\pm\infty} \phi(x) = \frac{\delta k}{R}$ and $\phi(0) \geq 0$ if and only if $k \geq k_1$. In both the cases $k \geq k_1$ and $k < k_1$, the two positive fixed points $x_2^\star$, $x_3^\star$ are created via fold bifurcation. □

As stated in Lemma 3.2, a k-value given by $k_1 \simeq 0.296588$ does exist such that if $k \geq k_1$ only simple dynamics is exhibited by the map (16). In fact $\phi'(x_i^\star) \geq 0$ for $i = 1, 2, 3$ so that the dynamic system is unable to produce cycles or more complex dynamics. The cases (a) and (b) discussed in Lemma 3.2 point 1 are presented in figure 1. In panel (a) the case with a unique non negative fixed point $x_1^\star$ is presented which attracts trajectories starting from every i.c. as $0 \leq \phi'(x_1^\star) < 1$. A different situation is shown

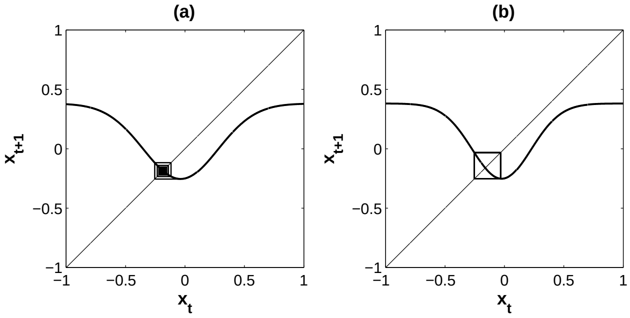

Figure 2: Koneig-Lamerary staircase diagram of map (16) for $k = 0.2 < k_1$, $R = 1.05$ and $\delta = 2$ with initial condition $x_0 = x_m$. (a) For $\beta = 4$, $x_1^\star$ is (locally) stable. (b) For $\beta = 8$, a locally stable two period cycle has been created.

in panel (b) concerning the case with three non negative fixed points: $A = x_1^\star$, $B = x_2^\star$ and $C = x_3^\star$ such that B is unstable while the other fixed points are (locally) stable. More precisely, let $(x_2^\star)_{-1}$ be the preimage of $x_2^\star$ different from $x_2^\star$, i.e. $\phi((x_2^\star)_{-1}) = x_2^\star$, then initial conditions belonging to $(-\infty, (x_2^\star)_{-1}) \cup (x_2^\star, +\infty)$ generate trajectories converging to the steady state C, while the set $((x_2^\star)_{-1}, x_2^\star)$ is the basin of attraction of the fixed point A. Observe that the case with three fixed points can emerge only for $k \in [k_1, k_1 + \epsilon)$ with ϵ sufficiently small, as if k is large enough, i.e. $k \to 1$, our system admits a unique positive globally stable steady state $\forall \beta, \delta$.

Consider now the case $k < k_1$. As stated in Lemma 3.2 point 2, also in this case two different scenarios may occur, i.e. one or three fixed points may exist. In both cases $x_1^\star < 0$ while the other fixed points $0 < x_2^\star \leq x_3^\star$, eventually owned by ϕ, belong to the increasing portion of ϕ so that $x_2^\star$ is unstable while $x_3^\star$ is locally stable and no more complicated features can be related to the positive steady states.

Let us go to focus on the study of the local dynamics around the fixed point $x_1^\star$. Let x_m denote the minimum point of our map, given by $x_m = \frac{1}{2}\delta(1 + 2k) - \frac{1}{4\beta}\sqrt{4\beta^2\delta^2(1 + 2k)^2 + 8\beta}$. If $x_m > \phi(x_m)$ then $x_1^\star < x_m$ and consequently $\phi'(x_1^\star) < 0$ so that complicated dynamics can be produced via period doubling bifurcations as some parameters are varied. Figure 2 confirms this fact.

In panel (a) the fixed point $x_1^\star$ is stable being $-1 < \phi'(x_1^\star) < 0$; in panel (b), as β is increased, the eigenvalue $\phi'(x_1^\star)$ has crossed -1 and, consequently,

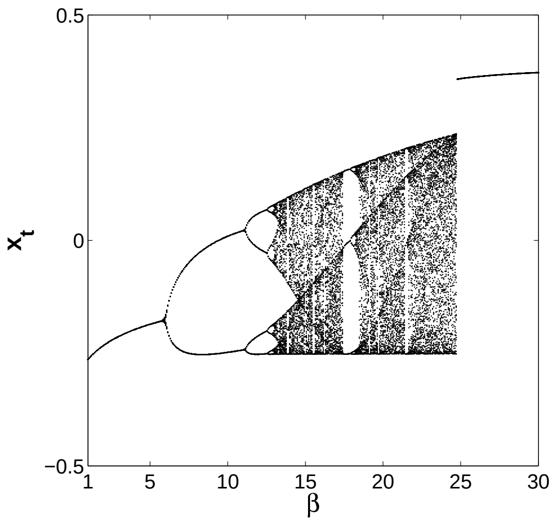

Figure 3: Bifurcation diagram of map (16) with respect to β for $k = 0.2 < k_1$, $R = 1.05$ and $\delta = 2$. The bifurcation cascade is presented.

the fixed point has lost its stability via period doubling bifurcation: a stable two period cycle has been created. We expect that several period doubling bifurcations are produced if the parameter β still increases. The bifurcation diagram in figure 3 confirms this fact and the period doubling route to chaos is exhibited.

Geometrical considerations and numerical simulations enable us to conclude that if k is small enough (i.e. $k < k_1$) and $x_1^\star < x_m$, the negative fixed point owned by ϕ may be stable or unstable. If it is unstable a more complex attractor exists, as cycles of period m or a more complex (even strange) invariant set. We denote with $\mathcal{A}$ the attractor of ϕ around $x_1^\star$. As previously underlined it may consist of the steady state, a period-m cycle or a more complex set.

We now consider the role of the parameters δ, β responsible for complicated dynamics, k being small enough. To reach this goal, it will be useful to consider the limiting cases $(\delta, \beta) \to (0, 0)$ and $\beta \to +\infty$, important in understanding the cases with small values of the couple (δ, β) and large but finite values of the parameter β. As previously stated, when β increases, more and more agents use the most sophisticated predictor, the fundamentalist belief system. In the extreme case $\beta \to +\infty$ all agents are fundamentalists, in the other extreme case $\beta \to 0$ no switching at all takes place and our model reduces to one with fixed and $k-$dependent proportions of agents. The next result describes what happens in these limiting cases.

Lemma 3.3. *(The limiting cases) Let $k \in (0, 1)$.*

1. *For* $(\delta, \beta) \to (0,0)$ *then* $x_0 = 0$ *is the unique, globally stable steady state.*

2. *For* $\beta \to +\infty$ *and* $\forall \delta > 0$*, there exists a unique globally stable steady state* $x^\star = \frac{\delta k}{R}$*.*

Proof. The results follow immediately from the limiting forms of $\phi(x, \delta, \beta)$:

$$\lim_{(\delta,\beta)\to(0,0)} \phi(x, \delta, \beta) = \frac{1}{R} e^{-(k+1)^2} x$$

and

$$\lim_{\beta\to+\infty} \phi(x, \delta, \beta) = \frac{\delta k}{R}.$$

□

As a consequence, as long as β and δ are both small enough, the system globally converges to a unique steady state given by the fundamental.

The dynamics is still simple for large but finite values of the parameter β as shown by the bifurcation diagram in figure 3 where the fixed point is globally stable $\forall \beta > \bar{\beta}$ being $\bar{\beta} \simeq 25$. This result holds for all $\delta > 0$. Notice that $\phi(x, \delta, \beta)$ diverges as δ goes to $+\infty$, so that we have to concentrate on intermediate values of the couple (δ, β) in order to investigate the cases in which $\mathcal{A}$ consists of more than one point (i.e. cycles or a more complex set). For this reason we focus on the two-dimensional bifurcation diagram as both δ and β vary taking into account small values of k.

The following figures 4, 5 and 6 contain cycle cartograms showing a two-parametric bifurcation diagram qualitatively. Each color represents a long-run dynamic behaviour for a given point in the parameter plane (δ, β) and for the initial condition $x_0 = x_m$. Observe a large variety of cycles of different orders. Also note that, as typical in one-dimensional bimodal dynamic maps, several period doubling and period halving cascades exist (see Hommes, 1994). These figures confirm the attained result, i.e. if $(\delta, \beta) \to (0,0)$ or $\beta \to +\infty$ the fixed point is globally stable.

We call *stability region* the set of points of the parameter plane (δ, β) for which the unique possible dynamics is the convergence to a steady state. In figures 4, 5 and 6 this set is represented by the blue region.

In figure 4 we present the bifurcation diagram with respect to δ and β with $k = 0.1$. A quite similar feature is observed with $k = 0.2$ as shown in figure 5 where the *stability region* is clearly increased. As a consequence we expect that the *stability region* decreases considering decreasing values of k.

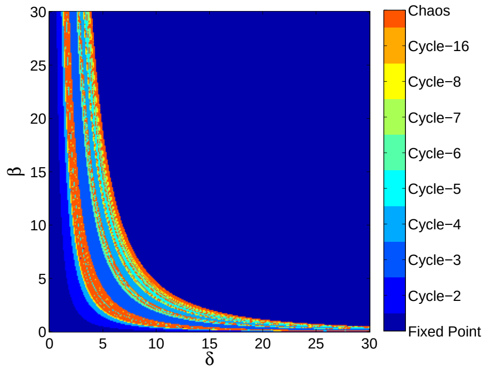

Figure 4: Cycle cartogram in (δ, β) plane for $k = 0.1$, $R = 1.05$ and i.c. $x_0 = x_m$.

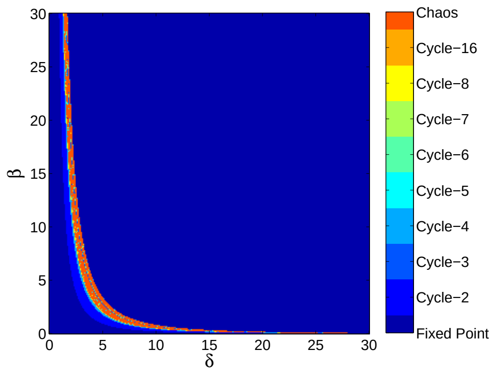

Figure 5: Cycle cartogram in (δ, β) plane for $k = 0.2$, $R = 1.05$ and i.c. $x_0 = x_m$.

Figure 6 confirms this reasoning: the two dimensional bifurcation diagram has been depicted for $k = 0.07$.

Notice that two curves exist in the parameter plane (δ, β), namely $\mathcal{C}_1$ and $\mathcal{C}_2$, such that the *stability region* is the union of two sets: the set of pairs (δ, β) below the curve $\mathcal{C}_1$ and that of points above $\mathcal{C}_2$. For couple of parameters between these two curves, the attractor $\mathcal{A}$ of the map has complicated features.

As we stressed, if $k < k_1$ and map ϕ has a unique steady state $x_1^\star$, every initial condition generates bounded sequences converging to the unique attractor $\mathcal{A}$, that may consist of the fixed point $x_1^\star$ or a more complex attrac-

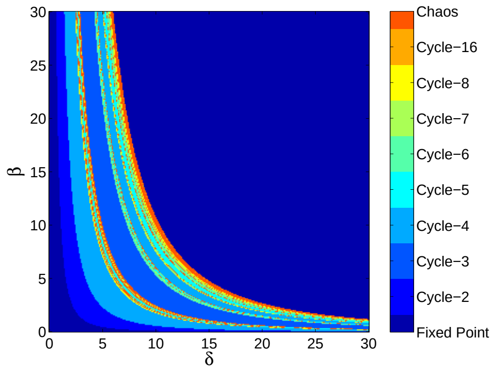

Figure 6: Cycle cartogram in (δ, β) plane for $k = 0.07$, $R = 1.05$ and i.c. $x_0 = x_m$.

tor (periodic or chaotic) located around $x_1^\star$.[2] However, according to Lemma 3.2, as some parameters vary, the map undergoes a fold bifurcation and two positive new steady states are created, $0 < x_2^\star < x_3^\star$, one stable and one unstable. Such cases are characterized by coexisting attractors: the attractor $\mathcal{A}$ and the stable steady state $x_3^\star$. As a consequence, the structure of the boundaries that separate basins of attraction of coexisting attractors becomes prominent. In fact, nonconnected basins of attraction may arise, due to the contact between critical points and basins' boundaries.[3]

In order to investigate the occurrence of *contact bifurcations*, i.e. global bifurcations responsible for the creation of basins with a fractal boundary, let us focus on the case $k < k_1$ with three fixed points, $x_1^\star < 0$ and $0 < x_2^\star < x_3^\star < \frac{\delta k}{R}$.

For certain parameter values, the graphical analysis of the map $\phi(x)$ shows that the basin of the attractor $\mathcal{A}$ is simply connected and bounded by the unstable fixed point $x_2^\star$ and its rank-1 preimage $(x_2^\star)_{-1} = \phi^{-1}(x_2^\star)$, i.e. $B(\mathcal{A}) = ((x_2^\star)_{-1}, x_2^\star)$. Initial conditions taken in the complementary of $B(\mathcal{A})$ generate trajectories converging to the stable steady state $x_3^\star$, so that the basin of $x_3^\star$ is formed by the union of two nonconnected portions: $B(x_3^\star) = (-\infty, (x_2^\star)_{-1}) \cup (x_2^\star, +\infty)$, where $B_1 = (x_2^\star, +\infty)$ is the immediate basin and $B_2 = (-\infty, (x_2^\star)_{-1}) = \phi(B_1)$ (see figure 7 panel (a)).

The situation drastically changes if the minimum value $\phi(x_m)$ decreases until it crosses the basin boundary $(x_2^\star)_{-1}$. In fact at $\phi(x_m) = (x_2^\star)_{-1}$ a contact

[2] See Devaney (1989).

[3] See Mira et al. (1996) and Abraham et al. (1997) for an extensive survive of the properties of noninvertible maps.

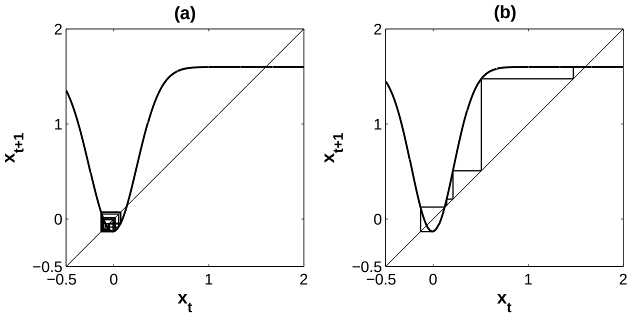

Figure 7: Basins of the attractor $\mathcal{A}$ for $k = 0.28 < k_1$, $R = 1.05$ and $\delta = 6$. (a) For $\beta = 8$, the trajectory starting from the i.c. $x_0 = x_m$ converges to $\mathcal{A}$. (b) For $\beta = 10$, the trajectory starting from the i.c. $x_0 = x_m$ converges to $x_3^\star$. A contact bifurcation occurred.

between the critical point and the basin boundary occurs, producing a global bifurcation which changes the structure of the basin. After this bifurcation, new preimages appear and they constitute an infinite and countable set of nonconnected portions and the generic trajectory converges to the stable steady state $x_3^\star$ (see figure 7 panel (b)). Notice that the complementary of the union of all these preimages is a Cantor set (with zero Lebesgue measure). As is well-known, the dynamics on the invariant Cantor set can be described by symbolic dynamics. More specifically, the dynamics of ϕ on the invariant Cantor set is equivalent to the dynamics of the chaotic shift map on the set of all one-sided symbolic sequences of 0's and 1's. This means that the dynamics of ϕ on the Cantor set is also chaotic.

4 Returning to the randomness

This section is devoted to the analysis of the stochastic dynamical system and the stability region, starting from the results reached in the deterministic study of the model.

4.1 The measure of the stability region

In this part of the development of the model, the relationship between the forecast of the fundamental value and the changing fundamentalist-chartist is explored. We fix $R = 1.05$, and we consider the same values of parameter k used to perform the simulations in the deterministic skeleton that are $k = 0.07$, $k = 0.1$ and $k = 0.2$ (see figures 4, 5 and 6). In the whole set of the cases, we have that the two parametric bifurcation diagram shows a *stability region* defined in the previous section, that can be described analytically by interpolating. Thus, we can state that the generic trajectory converges to a fixed point $\forall(\delta, \beta)$ belonging to Γ, with

$$\Gamma := \Big\{(\delta, \beta) \in (0, +\infty)^2 \,|\, \beta < f_1(\delta) \vee \beta > f_2(\delta)\Big\}, \tag{18}$$

where f_1, f_2 are continuous monotone deterministic transformations, obtained as best fit of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ coming out from the numerical analysis made in the previous section.[4]

We intend to provide an explicit formulation of an estimate of the measure of Γ. This analysis allows us to create a set of long-run strategies for each agent taking position in the market. Infact:

- the *stability region* is related to a kind of "complete information" about the future, in the sense that in the long term chaos or n-cycles equilibria are not allowed;

- the distributions of the stochastic parameters β and γ coincide with the asymptotic distributions of the stochastic processes β_t and γ_t. Therefore, the future is constructed by the investors, by choosing such stochastic parameters at the present time.

In order to proceed, we assume that δ follows a discrete distribution. We assume that δ takes values in a discrete and infinite set $D = \{\zeta_k\}_{k\in\mathbb{N}}$, where $\zeta_k \in (0, +\infty)$, for each k. The assumption on the support of δ lets the argumentation be clearer and it is not too restrictive, for two reasons: first, we can assume that $D \equiv \mathbb{Q} \cap (0, +\infty)$, that is dense in $(0, +\infty)$; moreover, δ is the expected value of the fundamental. Therefore, it is measured by currency, and it can be represented as a discrete variable.

By taking into account the best fit of the empirical bifurcation curves, $\mathcal{C}_1$ and $\mathcal{C}_2$ obtained in the analysis of the deterministic skeleton and described by the functions f_1 and f_2, we observe that a very good fit can be obtained basically by applying two fit functions types: a power law fit or a gaussian fit. In the second case, the fit-curve approximates better the empirical scatter plot if it is constructed as a sum of a relevant number of gaussian functions. Due to this fact, it is not reasonable to work by using a gaussian function fit, to provide a measure of Γ. Therefore, we rely on a power law.

[4]To this end, the Matlab® best fit tool has been used.

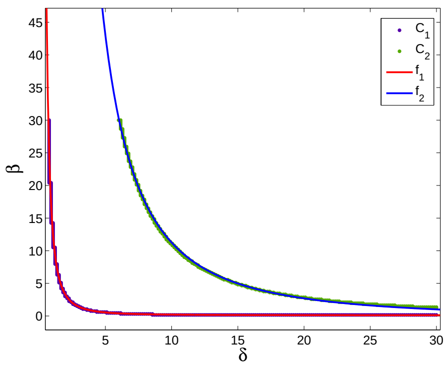

Figure 8: Best fit of the empirical bifurcation curves $\mathcal{C}_1$ and $\mathcal{C}_2$ for $k = 0.07$.

First we consider the case $k = 0.07$ (see figure 8). We fix an error of 10^{-6} in order to determine the approximating function f_1. The approximation scheme does not converge for f_2, and in this case we have to relax the error to 10^{-2}. In this case we obtain

$$f_1(\delta) = 13.61\delta^{-2.135} + 0.1082,$$

$$f_2(\delta) = 909.6\delta^{-1.889} - 0.475$$

and the scheme converges. Hence:

$$\Gamma = \Big\{(\delta, \beta) \in (0, +\infty)^2 \,|\, \beta < 13.61\delta^{-2.135} + 0.1082 \vee \beta > 909.6\delta^{-1.889} - 0.475\Big\}.$$

The probability measure of Γ is obtained by using standard stochastic calculus results. We have

$$P(\Gamma) = P(\{\delta > 0\} \cap [\{0 < \beta < 13.61\delta^{-2.135} + 0.1082\} \cup \{\beta > 909.6\delta^{-1.889} - 0.475\}]) =$$

$$= P(\{\delta > 0\} \cap \{0 < \beta < 13.61\delta^{-2.135} + 0.1082\}) +$$
$$+ P(\{\delta > 0\} \cap \{\beta > 909.6\delta^{-1.889} - 0.475\}) =$$

$$= \sum_{k=1}^{+\infty} P(\{0 < \beta < 13.61\delta^{-2.135} + 0.1082\} \cap \{\delta = \zeta_k\}) +$$

$$+ \sum_{k=1}^{+\infty} P(\{\beta > 909.6\delta^{-1.889} - 0.475\} \cap \{\delta = \zeta_k\}) =$$

$$= \sum_{k=1}^{+\infty} P(\{0 < \beta < 13.61\delta^{-2.135} + 0.1082\} \,\Big|\, \{\delta = \zeta_k\}) \cdot P(\delta = \zeta_k) +$$

$$+\sum_{k=1}^{+\infty} P(\{\beta > 909.6\delta^{-1.889} - 0.475\} \Big| \{\delta = \zeta_k\}) \cdot P(\delta = \zeta_k) =$$

$$= \sum_{k=1}^{+\infty} \Big\{ [P(0 < \beta < 13.61\zeta_k^{-2.135} + 0.1082) + P(\beta > 909.6\zeta_k^{-1.889} - 0.475)] \cdot P(\delta = \zeta_k) \Big\}.$$

By the knowledge of the probability distributions of the variables δ and β, we have an estimate of the probability of Γ.

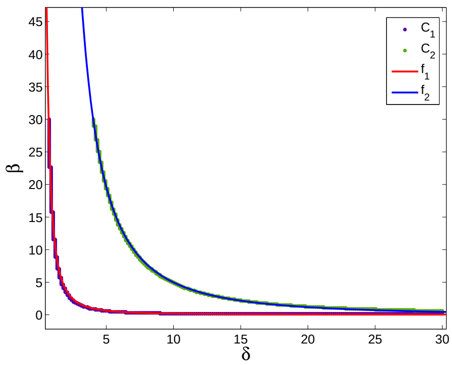

Figure 9: Best fit of the empirical bifurcation curves $\mathcal{C}_1$ and $\mathcal{C}_2$ for $k = 0.1$.

Analogously to the first case we assume $k = 0.1$ and we fix an error of 10^{-6} to approximate f_1 and 10^{-2} for f_2 (see figure 9). The explicit formulas are the following:

$$f_1(\delta) = 26.78\delta^{-1.95} - 0.08723,$$

$$f_2(\delta) = 451.3\delta^{-1.943} - 0.1728.$$

Then,

$$\Gamma = \Big\{ (\delta, \beta) \in (0, +\infty)^2 \,|\, \beta < 26.78\delta^{-1.95} - 0.08723 \vee \beta > 451.3\delta^{-1.943} - 0.1728 \Big\}.$$

The probability measure of Γ is obtained as in the previous case:

$$P(\Gamma) = \sum_{k=1}^{+\infty} \Big\{ [P(0 < \beta < 26.78\zeta_k^{-1.95} - 0.08723) + P(\beta > 451.3\zeta_k^{-1.943} - 0.1728)] \cdot P(\delta = \zeta_k) \Big\}.$$

Finally we consider $k = 0.2$. In this case, we fix an error of 10^{-6} to approximate either f_1 and f_2, since in both of the cases the best fit procedure converges (see figure 10). We obtain

$$f_1(\delta) = 26.78\delta^{-1.95} - 0.08719,$$

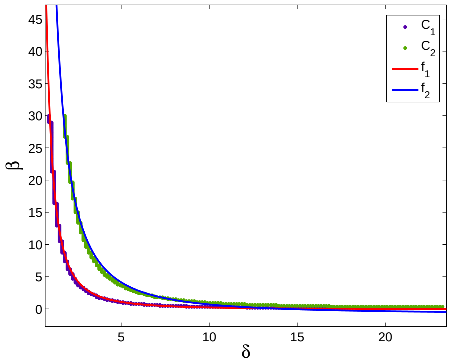

Figure 10: Best fit of the empirical bifurcation curves $\mathcal{C}_1$ and $\mathcal{C}_2$ for $k = 0.2$.

$$f_2(\delta) = 94.1\delta^{-2.024} + 0.5119.$$

Then,

$$\Gamma = \Big\{(\delta,\beta) \in (0,+\infty)^2 \,|\, \beta < 26.78\delta^{-1.95} - 0.08719 \vee \beta > 94.1\delta^{-2.024} + 0.5119\Big\}.$$

The probability measure of Γ is obtained as usual:

$$P(\Gamma) = \sum_{k=1}^{+\infty} \Big\{[P(0 < \beta < 26.78\zeta_k^{-1.95} - 0.08719) + P(\beta > 94.1\zeta_k^{-2.024} + 0.5119)] \cdot P(\delta = \zeta_k)\Big\}.$$

We obtain that, in the analyzed cases, the measure of the stability region depends on the shape of the probability distribution of δ. In particular, a simple analysis of the fit function shows that the measure of Γ grows as the distribution of δ is concentrated in small values.

4.2 A bayesian analysis: distribution of β

In this subsection we propose a method for choosing the belief rule by performing forecasts on the fundamental value. To this end, we analyze the model from a bayesian point of view and we do not explicate the values of k.

Our purpose is to provide a long-run equilibrium value for the fundamental, by estimating the evolution of δ_t and setting a limit as t goes to infinity, and then describing the factor β_t, that gives information on the heterogeneity and the explicit form of the agents' beliefs on the future prices.

The bifurcation curves are well approximated by using a sum of gaussian functions, as remarked in the previous subsection. Hence, from the knowledge of the value of the parameter δ, we derive the distribution of the parameter

β. We have an explicit approximate formula for the bounds of the region Γ as follows:

$$f_1(\delta) = \sum_{k=1}^{+\infty} a_{1,k}\exp[(-(\delta - b_{1,k})/c_{1,k})^2] \tag{19}$$

$$f_2(\delta) = \sum_{k=1}^{+\infty} a_{2,k}\exp[(-(\delta - b_{2,k})/c_{2,k})^2] \tag{20}$$

where $a_{j,k}$, $b_{j,k}$ and $c_{j,k}$ are opportunely chosen real numbers. Let us define the curves

$$\mathcal{C}_1 = \{\beta = f_1(\delta), \ \ \delta > 0\}, \tag{21}$$

$$\mathcal{C}_2 = \{\beta = f_2(\delta), \ \ \delta > 0\}. \tag{22}$$

If the value of δ is known, then β faces a probability density function that is a mixture of infinite gaussians, in both of the cases $\mathcal{C}_1$ and $\mathcal{C}_2$.

For each $t \in \mathbb{N}$, a couple of bifurcation curves $\mathcal{C}_{1,t}$ and $\mathcal{C}_{2,t}$ exist converging to the asymptotic curves defined in (21) and (22), as $t \to +\infty$. The following proposition states immediately:

Proposition 4.1. *Fix $t \in \mathbb{N}$. Let us assume that $\mathcal{C}_{1,t}$ and $\mathcal{C}_{2,t}$ are defined, respectively, by the functions $f_{1,t}$ and $f_{2,t}$, where*

$$f_{1,t} = \sum_{k=1}^{t} a_{1,k}\exp[(-(\delta - b_{1,k})/c_{1,k})^2],$$

$$f_{2,t} = \sum_{k=1}^{t} a_{2,k}\exp[(-(\delta - b_{2,k})/c_{2,k})^2].$$

Then

$$\lim_{t\to+\infty} f_{j,t} = f_j, \qquad j = 1, 2.$$

According to Proposition 4.1, if an agent in the market makes forecasts at time t on the long-run fundamental value δ, then she can calibrate her own adjustment parameter β_t by using a mixture of t gaussians.

Furthermore, since β_t is a mixture of gaussian distributions, then it can be written by an autoregressive simple rule:

$$\beta_t = \beta_{t-1} + \epsilon_t, \tag{23}$$

with $\epsilon_t \sim N(\mu_\epsilon(t), \sigma^2_\epsilon(t))$ and ϵ_{t_1} independent of ϵ_{t_2}, for each $t_1 \neq t_2$, $t_1, t_2 \in \mathbb{N}$. Moreover, the stability property of the gaussian distribution and the independence of ϵ's imply that $\beta_t \sim N(\mu_\beta(t), \sigma^2_\beta(t))$, with

$$\mu_\beta(t) = \sum_{k=1}^{t} \mu_\epsilon(k), \qquad \sigma^2_\beta(t) = \sum_{k=1}^{t} \sigma^2_\epsilon(k).$$

The adjustment parameter at time t can be obtained by the adjustment parameter at time $t-1$ added with a gaussian error, not necessarily with zero mean. Then, the proportion factor β_t satisfied a learning rule based on the previous values of this parameter. Such a rule is biased by a stochastic noise.

4.3 A bayesian analysis: the changing in persistence of the beliefs

In this subsection we explore the possible presence of a change in persistence of the investor's beliefs, on the boundary of the stability region. The change in persistence is a statistical concept related to the coexistence in the same time series of two different behaviors: stationarity with respect to a trend (in this case we have an integrated process of order 0, I(0)); presence of a stochastic trend (in this case we speak about an integrated process of order 1, I(1)). If a change in persistence occurs in a sample period $\{1, \dots, t\}$, then a breakpoint $\tau \in (0, 1)$ exists such that the time series β_k is I(0) (or I(1)), for $k = 1, \dots, [\tau t]$, and it is I(1) (or I(0)) for $k = [\tau t] + 1, \dots, t$, where $[\tau t]$ indicates the inferior integer part of τt. In our stochastic framework, this means that the change in fundamentalist-chartist proportion is ruled by a very different perception of the deviation from the fundamental: in the stationary case, the beliefs at different times show a common deterministic trend, and do not differ from each other; in the nonstationary case, the beliefs are driven by an increasing (or decreasing) parameter, and this implies a growing confidence in the deviation from the fundamental. Since we focus on the bifurcation curves $\mathcal{C}_{1,t}$ and $\mathcal{C}_{2,t}$, the switch due to a change in persistence is able to provide information about the stability region, and so about the steady states of the dynamical systems proposed in our framework.

Consider now a fixed value of δ. Then the corresponding β is a real number. Hence, we obtain that β_t shows a constant asymptotic behavior, as $t \to +\infty$. Statistically, this means that β_t is a stationary process at least in the last subsample of the sample period. If a change in persistence occurs, it is in the direction from I(1) to I(0) (figure 11 shows an example of the evolution of β_t in a sample period, fixing the value of δ).

To explore this phenomenon, we construct a model showing the case of change in persistence in the direction from I(1) to I(0). We rewrite β_k in line with (23) as follows:

$$\beta_k = \rho_k + \xi_k, \qquad k = 1, \dots, t, \tag{24}$$

where ξ_k and ρ_k are mutually independent i.i.d. gaussian processes with same

Figure 11: Evolution of β_t in a sample period $\{1, \ldots 53\}$, fixing the value of δ.

mean μ and variance σ^2, with

$$\mu = \frac{\mu_\beta(t)}{2t-1}, \qquad \sigma^2 = \frac{\sigma^2_\beta(t)}{2t-1}.$$

Now we introduce the presence of a stochastic error in the dynamic described by (24). We assume that ρ follows an evolution rule able to explore the eventual presence of a change in persistence in the dynamic of β. We have

$$\rho_k = \rho_{k-1} + 1(k \leq [\tau t])\eta_k, \qquad k = 1, \ldots t, \tag{25}$$

where $1(\cdot)$ is the indicator function, and η_k is an i.i.d. gaussian stochastic process that is mutually independent of ξ_k with mean zero and variance σ^2_η. The time-dependent parameter β_k yields a process which is nonstationary up to and including time $[\tau t]$ but it is I(0) after the break, if and only if $\sigma^2_\eta > 0$. In the converse case, we have that $\sigma^2_\eta = 0$, and the process is stationary in the whole sample period. In this case, a change in persistence does not occur. Formally, the statistical tests devoted to estimating the presence of change in persistence have null hypothesis $H_0 : \sigma^2_\eta = 0$ and alternative hypothesis $H_1 : \sigma^2_\eta > 0$. We suggest the statistics proposed by Kim (2000), Busetti and Taylor (2004) in order to explore the presence of a change in persistence.

4.4 A bayesian analysis: theoretical cluster analysis of the data

The basic idea of this last remark is to derive a theoretical methodology able to classify the parameter β_t as the value of δ changes. We aim to propose a clustering of β_t with respect to their ranges of variation, in the boundary of the stability region Γ. This analysis allows us to obtain information about the bifurcations of the dynamical system, in agreement with the deterministic study.

Let us fix $\delta = \bar{\delta}$, with $\bar{\delta} \in D$, where D is defined in subsection 4.1. For each $\bar{t} \in \mathbb{N}$, the fact that $\beta_{\bar{t}}$ is a gaussian distribution implies that it is uniquely determined by its mean and variance. We construct a 3-dimensional grid

$$G := \{(\delta_k, \mu_{k,t}, \sigma^2_{k,t}) \in D \times (0, +\infty)^2 \qquad \forall t, k \in \mathbb{N}\},$$

where $\mu_{k,t}$ and $\sigma^2_{k,t}$ are, respectively, the mean and the variance of β_t conditioned to $\delta = \delta_k$.

The mixture of gaussians model is a powerful estimation of the parameters μ's and σ^2's, and so of the distribution of the β's. The model assumes that the data are produced by a mixture of N multivariate gaussians.

The cluster classes we consider are related to the joint mean variance size of the β's, and they are obtained by considering a suitable partition of $(0, +\infty)^2$. In this way, we have information about how investors calibrate their beliefs by performing a forecast on the fundamental. The mean of the β's represents the degree of confidence on the fundamentals expected value (the biggest is the mean of β_t, the highest is the distortion of the fundamentals expected value at time $t+1$, given the information available at time t); the variance of the β's measures the stability of the distortion, and represents an error that the investor makes about the degree of confidence in the fundamental expected value. Such an error is proportional to the variance's size.

In our framework, the plots of the grid G are obtained by a mixture of gaussians. We suggest a couple of cluster procedures, involving mixture of gaussians data.

The first one is the standard clustering EM procedure (see Dempster et al., 1977) allowing to distinguish some classes within the grid G.

A more sophisticated cluster techniques is the Kohonen algorithm, based on the Voronoi tessellation of the grid G and on the introduction of some cluster weights. For details of the Kohonen network, we remind the reader of Kohonen (1989, 1991a, 1991b). A sufficient condition for the convergence of the algorithm in the multidimensional case can be found in Feng and Tirozzi (1997), Lin and Si (1998) and Sadeghi (2001).

5 Conclusions

This paper deals with an agent-based asset pricing model. Agents have heterogeneous beliefs and tend to imitate each others'. The imitative behavior is captured by introducing some stochastic parameters into the theoretical model. The dynamical map is bidimensional, and it exhibits randomness. We studied it by proving results related to the deterministic skeleton of the system and we proved that when most agents are chartists (fundamentalists) in type and the forecast of the fundamental provides small (high) values, only simple dynamics is possible and there is a *stability region* in the parameters' plane for which the only possible dynamics is convergence to a steady state. The system shows complex dynamics in the remaining cases.

The stability region was also analyzed by reintroducing stochasticity and we provided an explicit formula for its probability measure. Finally, we proposed a bayesian analysis, in order to explore the distribution of the adjustment term of the proportion of agents.

References

[1] Abraham, R., Gardini, L., Mira, C., 1997. Chaos in discrete dynamical systems (a visual introduction in two dimension), Springer-Verlag.

[2] Avery, C., Zemsky, P., 1998. Multidimensional Uncertainty and Herd Behavior in Financial Markets. *American Economic Review,* 88, (4), 724-748.

[3] Banerjee, A., 1992. A simple model of herd behavior. *Quarterly Journal of Economics*, 107, 787-818.
bibitemBHW92 Bikhchandani, S., Hirshleifer, D., Welch, I., 1992. A theory of fads, fashion, custom and cultural change as informational cascades. *Journal of Political Economy*, 100, 992-1027.

[4] Brock, W.A., Hommes, C.H., 1997. Rational route to randomness. *Econometrica,* 65, 1059-1095.

[5] Brock, W. A., Hommes, C. H., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of economic dynamics and control,* 22, 1235-1274.

[6] Busetti, F. and A.M.R., Taylor, 2004. Tests of stationarity against a change in persistence. *Journal of Econometrics* 123, 33-66.

[7] Chamley, C., Gale, D., 1994. Information revelation and strategic delay in a model of investment. *Econometrica*, 62, 1065-1086.

[8] Chari, V., Kehoe, P., 2000. Financial crises as herds. *Federal Reserve Bank of Minneapolis*, Working Paper 600.

[9] Conlisk, J., 1996. Why bounded rationality? *Journal of Economic Literature,* 34, 669-700.

[10] Dempster A.P., Laird N.M., and Rubin D.B., 1977. Maximum Likelihood from Incomplete Data via the EM algorithm *Journal of the Royal Statistical Society*, Series B, vol. 39, 1:1-38

[11] Devaney, R.L., 1989. An introduction to chaotic dynamical system. The Benjamin/Cummings Publishing Co, Menlo Park.

[12] Feng, J.F., Tirozzi B., 1997. Convergence Theorem for the Kohonen Feature Mapping Algorithms with VLRPs, *Computer Math. Applic.*, 33, 3, 45-63.

[13] Hommes, C. H., 1994. Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand. *Journal of Economic Behavior and Organization,* 24, 315-335.

[14] Hommes, C. H., 2001. Financial Markets as nonlinear adaptive evolutionary systems. *Quantitative Finance,* 1, 149-167.

[15] Hommes, C. H., 2006. Heterogeneous agent models in economics and finance. In *Handbook of Computational Economics,* L. Tesfatsion and K. L. Judd, eds., vol. 2, chap. 23. Elsevier, North Holland, 1109-1186.

[16] Kim, J.Y., 2000. Detection of change in persistence of a linear time series. *Journal of Econometrics* 95, 97-116.

[17] Kirman, A.P., 2006. Heterogeneity in economics. *Journal of Economic Interaction and Coordination,* 1, 89-117.

[18] Kohonen, T., 1989. Self-Organization and Associative Memory Process, (Springer-Verlag).

[19] Kohonen, T., 1991a. Analysis of a Simple Self-Organizing Process, *Biological Cybernetics*, 44, 135-140.

[20] Kohonen, T., 1991b. Self-Organizing maps: optimization approaches, *Artificial Neural Networks*, 1, 981-990.

[21] Lin, S., Si, J., 1998. Weight-Value Convergence of the SOM Algorithm for Discrete Input, *Neural Computation*, 10, 807-814.

[22] Mira, C., Gardini, L., Barugola, A., Cathala, J. C., 1996. Chaotic dynamics in two-dimensional noninvertible maps. World Scientific, Singapore.

[23] Sadeghi, A.A., 2001. Convergence in Distribution of the Multidimensional Kohonen Algorithm, *Journal of Applied Probability*, 38, 136-151.

[24] Simon, H., 1957. *Models of man.* John Wiley & Sons, New York.