



UNIVERSITÀ DEGLI STUDI DI MACERATA

Dipartimento di Istituzioni Economiche e Finanziarie

Global Attractors of Non-autonomous Difference Equations

David Cheban, Cristiana Mammana, Elisabetta Michetti

Quaderno di Dipartimento n. 47

Aprile 2008

Global Attractors of Non-autonomous Difference Equations

David Cheban, Cristiana Mammana, Elisabetta Michetti

Abstract

The article is devoted to the study of global attractors of quasi-linear non-autonomous difference equations, in particular we give the conditions for the existence of a compact global attractor. The obtained results are applied to the study of a triangular economic growth model $T : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ recently developed in Brianzoni S., Mammana C. and Michetti E. [1].

David Cheban, State University of Moldova.

E-mail: cheban@usm.md.

Cristiana Mammana, Università degli Studi di Macerata.

E-mail: mammana@unimc.it.

Elisabetta Michetti, Università degli Studi di Macerata.

E-mail: michetti@unimc.it.

1 Introduction

The global attractors play a very important role in the qualitative study of difference equations (both autonomous and non-autonomous). The present work is dedicated to the study of global attractors of quasi-linear non-autonomous difference equations

$$u_{n+1} = A(\sigma^n \omega)u_k + F(u_k, \sigma^n \omega), \quad (1)$$

where Ω is a metric space (generally speaking non-compact), $(\Omega, \mathbb{Z}_+, \sigma)$ is a dynamical system with discrete time \mathbb{Z}_+ , $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies to "the condition of smallness". Analogous problem has been studied in Cheban D. and Mammana C. [5] when the space Ω is compact.

The obtained results are applied to the study of a class of triangular maps $T = (T_1, T_2)$ describing an economic growth model in capital accumulation and population growth rate as recently proposed by Brianzoni S., Mammana C. and Michetti E. [1].¹

2 Triangular maps and non-autonomous dynamical systems

Let W and Ω be two complete metric spaces and denote by $X := W \times \Omega$ its Cartesian product. Recall that a continuous map $F : X \rightarrow X$ is called triangular if there are two continuous maps $f : W \times \Omega \rightarrow W$ and $g : \Omega \rightarrow \Omega$ such that $F = (f, g)$, i.e. $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$ for all $x = (u, \omega) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(u_n, \omega_n) \\ \omega_{n+1} = g(\omega_n), \end{cases} \quad (2)$$

for all $n \in \mathbb{Z}_+$, where \mathbb{Z}_+ is the set of all non-negative integer numbers.

Along with system (2) we consider the family of equations

$$u_{n+1} = f(u_n, g^n \omega) \quad (\omega \in \Omega), \quad (3)$$

¹The authors consider the neoclassical one-sector growth model with differential savings as in Bohm V. and Kaas L. [3], while assuming CES production function and the labour force dynamic described by the Beverton-Holt equation (see [2]), that has been largely studied in [6] and [7].

which is equivalent to system (2). Let $\varphi(n, u, \omega)$ be a solution of equation (3) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \rightarrow W$ ($(n, u, \omega) \mapsto \varphi(n, u, \omega)$) satisfies the following conditions:

- (i) $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;
- (ii) $\varphi(n + m, u, \omega) = \varphi(n, \varphi(m, u, \omega), \sigma(m, \omega))$ for all $n, m \in \mathbb{Z}_+, u \in W$ and $\omega \in \Omega$, where $\sigma(n, \omega) := g^n \omega$;
- (iii) the map $\varphi : \mathbb{Z}_+ \times W \times \Omega \rightarrow W$ is continuous.

Denote by $(\Omega, \mathbb{Z}_+, \sigma)$ the semi-group dynamical system generated by positive powers of map $g : \Omega \rightarrow \Omega$, i.e. $\sigma(n, \omega) := g^n \omega$ for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Recall [4, 8] that a triple $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ (or briefly φ) is called a cocycle over the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ with fiber W if the mapping $\varphi : \mathbb{Z}_+ \times W \times \Omega \rightarrow W$ possesses the properties (i)-(iii).

Let $X := W$ and (X, \mathbb{Z}_+, π) be a dynamical system on X , where $\pi(n, (u, \omega)) := (\varphi(n, u, \omega), \sigma(n, \omega))$ for all $u \in W$ and $\omega \in \Omega$, then (X, \mathbb{Z}_+, π) is called [8] a skew-product dynamical system, generated by the cocycle $\langle W, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$.

Taking into consideration this fact we can study triangular maps in the framework of cocycles with discrete time.

3 Global attractors of dynamical systems

Let \mathfrak{M} be some family of subsets from X and $\mathbb{T} = \mathbb{Z}_+$ or \mathbb{Z} .

Dynamical system (X, \mathbb{T}, π) is said to be \mathfrak{M} -dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi^t M \subseteq B(K, \varepsilon)$ for any $t \geq L(\varepsilon, M)$, where K is a certain fixed subset from X depending only on \mathfrak{M} . In this case K we will call the attractor for \mathfrak{M} .

For the applications the most important ones are the cases when K is bounded or compact and $\mathfrak{M} := \{\{x\} \mid x \in X\}$ or $\mathfrak{M} := C(X)$, or $\mathfrak{M} := \{B(x, \delta_x) \mid x \in X, \delta_x > 0\}$, or $\mathfrak{M} := B(X)$.

A dynamical system (X, \mathbb{T}, π) is called:

- point dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{t \rightarrow +\infty} \rho(xt, K) = 0; \quad (4)$$

- compact dissipative if the equality (4) takes place uniformly w.r.t. x on the compact subsets from X .

We denote by

$$J := \Omega(K) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi^\tau K},$$

then the set J does not depend of the choice of the attractor K and is characterized by the properties of the dynamical system (X, \mathbb{T}, π) . The set J is called a Levinson center of the dynamical system (X, \mathbb{T}, π) .

Theorem 3.1. [4] *Let (X, \mathbb{T}, π) be point dissipative. For (X, \mathbb{T}, π) to be compact dissipative it is necessary and sufficient that $\Sigma^+(K)$ be relatively compact for any compact $K \subseteq X$.*

Let E be a finite-dimensional Banach space and $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber E (or shortly φ).

A cocycle φ is called:

- dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t \rightarrow +\infty} |\varphi(t, u, \omega)| \leq r \quad (5)$$

for all $\omega \in \Omega$ and $u \in E$;

- uniform dissipative, if there exists a number $r > 0$ such that

$$\limsup_{t \rightarrow +\infty} \sup_{\omega \in \Omega', |u| \leq R} |\varphi(t, u, \omega)| \leq r$$

for all compact subset $\Omega' \subseteq \Omega$ and $R > 0$.

Let (X, \mathbb{T}, π) be a dynamical system and $x \in X$. Denote by $\omega_x := \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, x)}$ the ω -limit set of point x .

Theorem 3.2. *The following statements hold:*

- (i) *if the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ and the cocycle φ are point dissipative, then the skew-product dynamical system (X, \mathbb{Z}_+, π) is point dissipative;*
- (ii) *if the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is compact dissipative and the cocycle φ is uniform dissipative, then the skew-product dynamical system (X, \mathbb{Z}_+, π) is compact dissipative.*

Proof. Let $x := (u, \omega) \in X := E \times \Omega$, then under the conditions of Theorem the set $\Sigma_x := \{\pi(t, x) : t \in \mathbb{Z}_+\}$ is relatively compact and $\omega_x \subseteq B[0, r] \times K$, where $B[0, r] := \{u \in E : |u| \leq r\}$, r is a number figuring in the inequality (5) and K is a compact appearing in (4). Thus the dynamical system (X, \mathbb{Z}_+, π) is point dissipative.

According to first statement of Theorem the skew-product dynamical system (X, \mathbb{Z}_+, π) is point dissipative. Let M be an arbitrary compact subset from $X := E \times \Omega$, then there are $R > 0$ and a compact subset $\Omega' \subseteq \Omega$ such that $M \subseteq B[0, R] \times \Omega'$. Note that $\Sigma_M^+ := \{\pi(t, M) : t \in \mathbb{Z}_+\} \subseteq \Sigma_{B[0, R] \times \Omega'}^+ := \{(\varphi(t, u, \omega), \sigma(t, \omega)) : t \in \mathbb{Z}_+, u \in B[0, R], \omega \in \Omega'\}$. We will show that the set Σ_M^+ is relatively compact. In fact, let $\{x_k\} \subseteq \Sigma_M^+$, then there are $\{u_k\} \subseteq B[0, R]$, $\{\omega_k\} \subseteq \Omega'$ and $\{t_k\} \subseteq \mathbb{Z}_+$ such that $x_k = (\varphi(t_k, u_k, \omega_k), \sigma(t_k, \omega_k))$. By compact dissipativity of dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ and uniform dissipativity of the cocycle φ the sequences $\{\varphi(t_k, u_k, \omega_k)\}$ and $\{\sigma(t_k, \omega_k)\}$ are relatively compact and, consequently, the sequence $\{x_k\}$ is so. Now to finish the proof it is sufficient to refer to Theorem 3.1. \square

4 Global attractors of quasi-linear triangular systems

Consider a difference equation

$$u_{n+1} = f(u_n, \sigma^n \omega) \quad (\omega \in \Omega). \quad (6)$$

Denote by $\varphi(n, u, \omega)$ a unique solution of equation (6) with the initial condition $\varphi(0, u, \omega) = u$.

Equation (6) is said to be dissipative (respectively, uniform dissipative), if there exists a positive number r such that

$$\limsup_{n \rightarrow +\infty} |\varphi(n, u, \omega)| \leq r \quad (\text{respectively, } \limsup_{n \rightarrow +\infty} \sup_{\omega \in \Omega', |u| \leq R} |\varphi(n, u, \omega)| \leq r)$$

for all $u \in E$ and $\omega \in \Omega$ (respectively, for all $R > 0$ and $\Omega' \in C(\Omega)$).

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma^n \omega)u_k + F(u_k, \sigma^n \omega), \quad (7)$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness".

Denote by $U(k, \omega)$ the Cauchy matrix for the linear equation

$$u_{n+1} = A(\sigma^n \omega)u_k.$$

Theorem 4.1. *Suppose that the following conditions hold:*

(i) *there are positive numbers N and $q < 1$ such that*

$$\|U(n, \omega)\| \leq Nq^n \quad (n \in \mathbb{Z}_+); \quad (8)$$

(ii) *$|F(u, \omega)| \leq C + D|u|$ ($C \geq 0$, $0 \leq D < (1 - q)N^{-1}$) for all $u \in E$ and $\omega \in \Omega$.*

Then equation (7) is uniform dissipative and

$$|\varphi(n, u, \omega)| \leq (q + DN)^{n-1}qN|u| + \frac{CN}{q-1}(q^{n-1} - 1). \quad (9)$$

Proof. This statement can be proved using the same type of arguments as in the proof of Theorem 5.2 from [5] and we omit the details. \square

Let $\langle E, \varphi, (\Omega, \mathbb{Z}_+, \sigma) \rangle$ be a cocycle over $(\Omega, \mathbb{Z}_+, \sigma)$ with the fiber E .

Theorem 4.2. *Let $(\Omega, \mathbb{Z}_+, \sigma)$ be a compact dissipative dynamical system and φ be a cocycle generated by equation (7). Under the conditions of Theorem 4.1 the skew-product dynamical system (X, \mathbb{Z}_+, π) , generated by cocycle φ admits a compact global attractor.*

Proof. This statement follows directly from Theorems 4.1 and 3.2. \square

Theorem 4.3. *Let $A \in C(\Omega, [E])$ and $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

(i) *the dynamical system $(\Omega, \mathbb{Z}_+, \sigma)$ is compact dissipative and J_Ω its Levinson center;*

(ii) *there exist positive numbers N and $q < 1$ such that inequality (8) holds;*

(iii) *there exists $C > 0$ such that $|F(0, \omega)| \leq C$ for all $\omega \in \Omega$;*

(iv) *$|F(u_1, \omega) - F(u_2, \omega)| \leq L|u_1 - u_2|$ ($0 \leq L < N^{-1}(1 - q)$) for all $\omega \in \Omega$ and $u_1, u_2 \in E$.*

Then

(i) *the equation (7) (the cocycle φ generated by this equation) admits a compact global attractor;*

(ii) *there are two positive constants \mathcal{N} and $\nu < 1$ such that*

$$|\varphi(n, u_1, \omega) - \varphi(n, u_2, \omega)| \leq \mathcal{N}\nu^n|u_1 - u_2| \quad (10)$$

for all $u_1, u_2 \in E$ and $n \in \mathbb{Z}_+$.

Proof. This statement can be proved by slight modification the proof of Theorem 5.9 from [5] and we omit the details. \square

5 Non-Autonomous Dynamical Systems with Convergence

A cocycle φ over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W is called compactly dissipative if the skew-product dynamical system (X, \mathbb{T}_1, π) associated by cocycle φ ($X := W \times Y$ and $\pi := (\varphi, \sigma)$) is so.

$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is said to be convergent if the following conditions are valid:

- (i) the dynamical systems (X, \mathbb{T}_1, π) and $(Y, \mathbb{T}_2, \sigma)$ are compactly dissipative;
- (ii) the set $J_X \cap X_y$ contains at most one point for all $y \in J_Y$, where $X_y := h^{-1}(y) := \{x | x \in X, h(x) = y\}$ and J_X (respectively, J_Y) is the Levinson center of the dynamical system (X, \mathbb{T}_1, π) (respectively, $(Y, \mathbb{T}_2, \sigma)$).

Theorem 5.1. [4, Ch.II] *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be a non-autonomous dynamical system and the following conditions be fulfilled:*

- (i) *the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is compact dissipative and J_Y its Levinson center;*
- (ii) *there exists a homomorphism γ from $(Y, \mathbb{T}_2, \sigma)$ to (X, \mathbb{T}_1, π) such that $h \circ \gamma = Id_Y$;*
- (iii) *$\lim_{t \rightarrow +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ for all $x_1, x_2 \in X$ ($h(x_1) = h(x_2)$).*

Then

- (i) *the dynamical system (X, \mathbb{T}_1, π) is compactly dissipative and $\gamma(J_Y) = J_X$;*
- (ii) *J_y consists a single point $\gamma(y)$ for all $y \in J_Y$.*

Theorem 5.2. *Let $A \in C(\Omega, [E])$ and $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

- (i) *the dynamical system $(\Omega, \mathbb{Z}, \sigma)$ is compact dissipative and J_Ω its Levinson center;*
- (ii) *there exist positive numbers N and $q < 1$ such that inequality (8) holds;*
- (iii) *there exists $C > 0$ such that $|F(0, \omega)| \leq C$ for all $\omega \in \Omega$;*

(iv) $|F(u_1, \omega) - F(u_2, \omega)| \leq L|u_1 - u_2|$ ($0 \leq L < N^{-1}(1 - q)$) for all $\omega \in \Omega$ and $u_1, u_2 \in E$.

Then

(i) the equation (7) (the cocycle φ generated by this equation) admits a compact global attractor $\{I_\omega \mid \omega \in J_\Omega\}$ and I_ω consists a single point u_ω (i.e. $I_\omega = \{u_\omega\}$) for all $\omega \in J_\Omega$;

(ii) the mapping $\omega \mapsto u_\omega$ is continuous and $\varphi(t, u_\omega, \omega) = u_{\sigma(t, \omega)}$ for all $\omega \in J_\Omega$ and $t \in \mathbb{Z}$;

(iii) there are two positive constants \mathcal{N} and $\nu < 1$ such that

$$|\varphi(n, u_1, \omega) - \varphi(n, u_2, \omega)| \leq \mathcal{N}\nu^n |u_1 - u_2| \quad (11)$$

for all $u_1, u_2 \in E$ and $n \in \mathbb{Z}_+$;

(iv)

$$|\varphi(n, u, \omega) - u_{\sigma^n \omega}| \leq \mathcal{N}\nu^n |u - u_\omega| \quad (12)$$

for all $u \in E$, $\omega \in J_\Omega$ and $n \in \mathbb{Z}_+$.

Proof. Let $\langle E, \varphi, (\Omega, \mathbb{Z}, \sigma) \rangle$ be the cocycle generated by equation (7) and $C_b(\Omega, E)$ be the space of all continuous and bounded functions $\mu : \Omega \mapsto E$ equipped with the sup-norm. For every $n \in \mathbb{Z}_+$ we define the mapping $S^n : C_b(\Omega, E) \mapsto C_b(\Omega, E)$ by equality $(S^n \mu)(\omega) := \varphi(n, \mu(\sigma(-n, \omega)), \sigma(-n, \omega))$ for all $\omega \in \Omega$. It easy to verify that the family of mappings $\{S^n \mid n \in \mathbb{Z}_+\}$ forms a commutative semigroup. From the inequality (9) it follows that $S^n \mu \in C_b(\Omega, E)$ for every $\mu \in C_b(\Omega, E)$ and $n \in \mathbb{Z}_+$. On the other hand from the inequality (10) we have

$$\|S^n \mu_1 - S^n \mu_2\| \leq \mathcal{N}\nu^n \|\mu_1 - \mu_2\|$$

for all $\mu_1, \mu_2 \in C_b(\Omega, E)$ and $n \in \mathbb{Z}_+$, where $\mathcal{N} := \frac{qN}{q+LN}$ and $\nu := q + LN$. Under the conditions of Theorem $\nu = q + LN < q + 1 - q = 1$ and, consequently, the semi-group $\{S^n \mid n \in \mathbb{Z}_+\}$ is contracting. Thus there exists a unique fixed point $\mu \in C_b(\Omega, E)$ of the semi-group $\{S^n \mid n \in \mathbb{Z}_+\}$ and hence

$$\mu(\sigma(n, \omega)) = \varphi(n, \mu(\omega), \omega)$$

for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Let $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}, \sigma), h \rangle$ be the non-autonomous dynamical system associated by cocycle φ (i.e. $X := E \times \Omega$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \mapsto \Omega$). Under the conditions of Theorem by Theorem 4.3 we have $\rho(x_1 t, x_2 t) \leq$

$\mathcal{N}e^{-\nu t}\rho(x_1, x_2)$ for all $x_1, x_2 \in X$ ($h(x_1) = h(x_2)$). Since $\gamma := (\mu, Id_\Omega)$ is an invariant section of the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (\Omega, \mathbb{Z}, \sigma), h \rangle$, then according to Theorem 5.1 the dynamical system (X, \mathbb{Z}_+, π) is compactly dissipative, its Levinson center $J_X = \gamma(J_\Omega)$ and $J_\omega := J \cap X_\omega$ ($X_\omega := h^{-1}(\omega)$) consists a single point $\gamma(\omega)$, i.e. $J_\omega = \{\gamma(\omega)\}$ for all $\omega \in \Omega$. Taking into consideration that the skew-product dynamical system (X, \mathbb{Z}_+, π) is compact dissipative, $J_\omega = I_\omega \times \omega$ and $\gamma = (\mu, Id_\Omega)$ we obtain $I_\omega = \mu(\omega)$ for all $\omega \in J_\Omega$. \square

6 Economic Application

6.1 The model

Dynamic economic growth models have often considered the standard, one-sector neoclassical Solow model (see Solow S. R. [9]). Bohm V. and Kaas L. [3] considered the role of differential savings behavior between workers and shareholders and its effects with regard to stability of stationary steady states within the framework of the discrete-time Solow growth model. More recently, Brianzoni S., Mammana C. and Michetti E. [1] proposed a discrete-time version of the Solow growth model with differential savings as formalized by Bohm V. and Kaas L. [3] while considering two different assumptions. Firstly they assume the CES production function. Secondly they assume the labor force growth rate not being constant, in particular they consider a model for density dependent population growth described by the Beverton-Holt equation (see [2]).

The resulting system (T, \mathbb{R}_+^2) describing capital accumulation k and population n dynamics of the model studied in Brianzoni S., Mammana C. and Michetti E. [1], where $T = (T_1, T_2)$, is given by

$$T_1(k, n) = \frac{(1 - \delta)k + (k^\rho + 1)^{\frac{1-\rho}{\rho}} (s_w + s_r k^\rho)}{1 + n}$$

and

$$T_2(n) = \frac{rhn}{h + (r - 1)n}$$

for all $(k, n) \in \mathbb{R}_+^2$. In the model, $\delta \in (0, 1)$ is the depreciation rate of capital, $s_w \in (0, 1)$ and $s_r \in (0, 1)$ are the constant saving rates for workers and shareholders respectively,² $\rho \in (-\infty, 1)$, $\rho \neq 0$ is a parameter related to the

²The authors also assume $s_w \neq s_r$ since the standard growth model of Solow R. M. [9] is obtained if the two savings propensities are equal.

elasticity of substitution between the production factors given by $1/(1 - \rho)$, $h > 0$ is the carrying capacity (for example resource availability) and $r > 1$ is the inherent growth rate (such a rate is determined by life cycle and demographic properties such as birth rates etc.). The Beverton-Holt T_2 have been studied extensively in Cushing J. M. and Henson S. M. [6, 7].

6.2 Invariant sets

Invariant sets of the mapping $T : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$.

Lemma 6.1. *The following sets are invariant for the mapping T :*

- (1) $A_1 = \{(k, 0) : k \in \mathbb{R}_+\}$
- (2) $A_2 = \{(k, h) : k \in \mathbb{R}_+\}$
- (3) $A_3 = \{(k, n) : 0 < n < h, k \in \mathbb{R}_+\}$
- (4) $A_4 = \{(k, n) : n > h, k \in \mathbb{R}_+\}$

Proof. This statement follows from the fact that the system T is triangular and the sets: $B_1 = \{0, 0\}$, $B_2 = \{0, h\}$, $B_3 = \{(0, n) | 0 < n < h\}$ and $B_4 = \{(0, n) | h < n\}$ are invariant with respect to one dimensional map $T_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. \square

Remark 6.2. *If $\rho \in (-\infty, 0)$, then*

- (i) $T_1(0, n) = 0$ for all $n \in \mathbb{R}_+$;
- (ii) T admits also the 5th invariant set $A_5 = \{(0, n) : n \in \mathbb{R}_+\}$.

6.3 Existence of an attractor for $\rho \in (-\infty, 0)$.

Theorem 6.3. *If $\rho < 0$, then the dynamical system (\mathbb{R}_+^2, T) admits a compact global attractor.*

Proof. Assume $\rho \in (-\infty, 0)$ and let $\lambda = -\rho$, then $\lambda \in (0, +\infty)$. We write T_1 in terms of λ

$$\begin{aligned} T_1(k, n) &= \frac{1}{1+n} \left[(1-\delta)k + (k^{-\lambda} + 1)^{\frac{1+\lambda}{-\lambda}} (s_w + s_r k^{-\lambda}) \right] = \\ &= \frac{1}{1+n} \left[(1-\delta)k + \left(\frac{1+k^\lambda}{k^\lambda} \right)^{-\frac{1+\lambda}{\lambda}} \left(\frac{s_r + s_w k^\lambda}{k^\lambda} \right) \right] = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1+n} \left[(1-\delta)k + \left(\frac{k^\lambda}{1+k^\lambda} \right)^{\frac{1+\lambda}{\lambda}} \left(\frac{s_r + s_w k^\lambda}{k^\lambda} \right) \right] = \\
&= \frac{1}{1+n} \left[(1-\delta)k + \frac{k}{(1+k^\lambda)^{\frac{1+\lambda}{\lambda}}} (s_r + s_w k^\lambda) \right] = \\
&= \frac{1}{1+n} \left[(1-\delta)k + \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} \frac{s_r + s_w k^\lambda}{1+k^\lambda} \right]. \tag{13}
\end{aligned}$$

Note that $\frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} \rightarrow 1$ as $k \rightarrow +\infty$, $\frac{s_r + s_w k^\lambda}{1+k^\lambda} \rightarrow s_w$ as $k \rightarrow +\infty$ and, consequently, there exists $M > 0$ such that

$$\left| \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} \frac{s_r + s_w k^\lambda}{1+k^\lambda} \right| \leq M, \tag{14}$$

for all $k \in [0, +\infty)$.

Since $0 \leq \frac{1}{1+n} \leq 1$ for all $n \in \mathbb{R}_+$, then from (13) and (14) we obtain

$$0 \leq T_1(k, n) \leq \alpha k + M \tag{15}$$

for all $n, k \in \mathbb{R}_+$, where $\alpha := 1 - \delta > 0$.

Since the map T is triangular, to prove this theorem it is sufficient to apply Theorem 4.2. Theorem is proved. \square

Remark 6.4. 1. It is easy to see that the previous theorem is true also for $\delta = 1$ because in this case $\alpha = 1 - \delta = 0$ and from (15) we have $T_1(k, n) \leq M$, $\forall k, n \in \mathbb{R}_+$. Now it is sufficient to refer to Theorem 3.2.

2. If $\delta = 0$ the problem is open.

According to Theorem 6.3, it is possible to conclude that if the elasticity of substitution between the two production factors (capital and labour) is positive and lesser than one (that is $\rho < 0$), capital and population dynamics cannot be explosive so economic patterns are bounded.

6.4 Existence of an attractor for $\rho \in (0, 1)$ and $s_r < \delta$.

The dynamical system (X, \mathbb{T}, π) we will call:

- locally completely continuous if for every point $p \in X$ there exist $\delta = \delta(p) > 0$ and $l = l(p) > 0$ such that $\pi^l B(p, \delta)$ is relatively compact;
- weakly dissipative if there exist a nonempty compact $K \subseteq X$ such that for every $\varepsilon > 0$ and $x \in X$ there is $\tau = \tau(\varepsilon, x) > 0$ for which $x\tau \in B(K, \varepsilon)$. In this case we will call K weak attractor.

Note that every dynamical system (X, \mathbb{T}, π) defined on the locally compact metric space X is locally completely continuous.

Theorem 6.5. [4] *For the locally completely continuous dynamical systems the weak, point and compact dissipativity are equivalent.*

Theorem 6.6. *If $\rho \in (0, 1)$ and $s_r < \delta$, then the mapping T admits a compact global attractor.*

Proof. If $\rho \in (0, 1)$ and $k > 0$ we have

$$\begin{aligned} T_1(k, n) &= \frac{1}{1+n} \left[(1-\delta)k + (k^\rho + 1)^{\frac{1-\rho}{\rho}} (s_w + s_r k^\rho) \right] = \\ &= \frac{1}{1+n} \left[(1-\delta)k + \frac{(k^\rho + 1)^{\frac{1}{\rho}}}{1+k^\rho} (s_w + s_r k^\rho) \right] = \\ &= \frac{1}{1+n} [(1-\delta)k + s_r k + \theta(k)k] \end{aligned} \quad (16)$$

where $\theta(k) := \frac{(k^\rho+1)^{\frac{1}{\rho}}}{k(1+k^\rho)}(s_w + s_r k^\rho) - s_r \rightarrow 0$ as $k \rightarrow +\infty$. In fact $\frac{(k^\rho+1)^{\frac{1}{\rho}}}{k} \rightarrow 1$ as $k \rightarrow +\infty$ while $\frac{(s_w+s_r k^\rho)}{1+k^\rho} \rightarrow s_r$ as $k \rightarrow +\infty$ and, consequently,

$$\frac{\frac{(k^\rho+1)^{\frac{1}{\rho}}}{1+k^\rho}(s_w + s_r k^\rho)}{s_r k} = \frac{(k^\rho + 1)^{\frac{1}{\rho}}}{k} \frac{(s_w + s_r k^\rho)}{s_r(k^\rho + 1)} \rightarrow 1$$

as $k \rightarrow +\infty$, i.e. $\frac{(k^\rho+1)^{\frac{1}{\rho}}}{1+k^\rho}(s_w + s_r k^\rho) = s_r k + \theta(k)k$. From (16) we have

$$T_1(k, n) = \frac{1}{1+n} [(1-\delta + s_r)k + \theta(k)k]$$

for all $(k, n) \in \mathbb{R}_+^2$ with $k > 0$.

Since $s_r < \delta$ then $\alpha := 1 - \delta + s_r < 1$. Let $R_0 > 0$ be a positive number such that

$$|\theta(k)| < \frac{1-\alpha}{2}, \quad (17)$$

for all $k > R_0$. Note that for every $(k_0, n_0) \in \mathbb{R}_+^2$, with $k_0 > R_0$, the trajectory $\{T^t(k, n) \mid t \in \mathbb{Z}_+\}$ starting from point (k_0, n_0) at the initial moment $t = 0$, at least one time intersects the compact $K_0 := [0, h_0] \times [0, R_0]$, ($h_0 > h$). In fact, if we suppose that this statement is false, then there exists a point $(k_0, n_0) \in \mathbb{R}_+^2 \setminus K_0$ such that

$$(k_t, n_t) := T^t(k_0, n_0) \in \mathbb{R}_+^2 \setminus K_0 \quad (18)$$

for all $t \in \mathbb{Z}_+$. Taking into consideration that $n_t \rightarrow h$ (or 0) as $t \rightarrow +\infty$, we obtain from (18) that $k_t > R_0$ for all $t \geq t_0$, where t_0 is a sufficiently large number from \mathbb{Z}_+ . Without loss of generality, we may suppose that $t_0 = 0$ (if $t_0 > 0$ then we start from the initial point $(n_{t_0}, k_{t_0}) := T^{t_0}(n_0, k_0)$, where $T^{t_0} := T \circ T^{t_0-1}$ for all $t_0 \geq 2$). Thus we have

$$k_t > R_0 \quad (19)$$

for all $t \geq 0$ and

$$k_{t+1} = \frac{1}{1+n} [\alpha k_t + \theta(k_t)k_t] \quad (20)$$

From (17) and (20) we obtain

$$k_{t+1} \leq \alpha k_t + \frac{1-\alpha}{2} k_t = \frac{1+\alpha}{2} k_t \quad (21)$$

since $\frac{1}{1+n} \leq 1$ for all $t \geq 0$. From (21) we have

$$k_t \leq \left(\frac{1+\alpha}{2}\right)^t k_0 \rightarrow 0 \text{ as } t \rightarrow +\infty, \quad (22)$$

but (19) and (22) are contradictory. The obtained contradiction proves the statement. Let now $(k_0, n_0) \in \mathbb{R}_+^2$ be an arbitrary point.

- (a) If $k_0 < R_0$ and $k_t \leq R_0$ for all $t \in \mathbb{N}$, then $\limsup_{t \rightarrow +\infty} k_t \leq R_0$;
- (b) If there exists $t_0 \in \mathbb{N}$ such that $k_{t_0} > R_0$, then there exists $\tau_0 \in \mathbb{N}$ ($\tau_0 > t_0$) such that $(k_{\tau_0}, n_{\tau_0}) \in K_0$ (see the proof above).

Thus we proved that for all $(k_0, n_0) \in \mathbb{R}_+^2$ there exists $\tau_0 \in \mathbb{N}$ such that $(k_{\tau_0}, n_{\tau_0}) \in K_0$. According to Theorem 6.5 the dynamic system (\mathbb{R}_+^2, T) admits a compact global attractor. The theorem is proved. \square

6.5 Structure of the attractor

A fixed point $p \in X$ of dynamical system (X, \mathbb{T}, π) is called

- Lyapunov stable if for arbitrary positive number $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, p) < \delta$ implies $\rho(\pi(t, x), p) < \varepsilon$ for all $t \geq 0$;
- attracting if there exists $\delta_0 > 0$ such that $\lim_{t \rightarrow +\infty} \rho(\pi(t, x), p) = 0$ for all $x \in B(p, \delta_0) := \{x \in X \mid \rho(x, p) < \delta_0\}$;
- asymptotically stable if it is Lyapunov stable and attracting.

Theorem 6.7. *Suppose that $\rho < 0$ and one of the following conditions hold:*

- (i) $s_w < \min\{\delta, s_r\}$ and $0 < \lambda < \lambda_0$, where λ_0 is a positive root of the quadratic equation $(s_r - s_w)\lambda^2 + (s_r - 2\delta)\lambda - \delta = 0$;
- (ii) $s_r < s_w < \delta$.

Then

- (i) the dynamic system (\mathbb{R}_+^2, T) admits a compact global attractor $J = \{(0, n) \mid 0 \leq n \leq h\}$;
- (ii) for all point $x := (k, n) \in \mathbb{R}_+^2$ with $n > 0$ the ω -limit set ω_x of x consists a single fixed point $(0, h)$ of dynamical system (\mathbb{R}_+^2, T) ;
- (iii) the fixed point $(0, h)$ is asymptotically stable.

Proof. Assume $\rho \in (-\infty, 0)$ and let $\lambda = -\rho$, then $\lambda \in (0, +\infty)$. We write T_1 in terms of λ (see the proof of Theorem 6.6)

$$T_1(k, n) = \frac{1}{1+n} \left[(1-\delta)k + \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} \frac{s_w + s_r k^\lambda}{1+k^\lambda} \right].$$

Denote by

$$f(k) := \frac{k}{(1+k^\lambda)^{\frac{1}{\lambda}}} \frac{s_w + s_r k^\lambda}{1+k^\lambda},$$

then

$$f'(k) = \frac{s_w + (-s_w\lambda + (\lambda+1)s_r)k^\lambda}{(1+k^\lambda)^{2+1/\lambda}}.$$

It easy to verify that under the conditions of Theorem $f'(k) < s_w$ for all $k \geq 0$. Consider the non-autonomous difference equation

$$k_{t+1} = A(\sigma(t, n))k_t + F(k_t, \sigma(t, n)) \quad (23)$$

corresponding to triangular map $T = (T_1, T_2)$, where $A(n) := \frac{1}{n+1}$, $F(k, n) := \frac{1}{n+1}f(k)$ and $\sigma(t, n) := T_2^t(n)$ for all $t \in \mathbb{Z}_+$ and $n \in \mathbb{R}_+$. Under the conditions of Theorem we can apply Theorem 5.2. By this Theorem the dynamical system (\mathbb{R}_+^2, T) is compact dissipative with Levinson center J and there exists a unique continuous bounded function $\mu : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $J = \{(\mu(n), n) \mid n \in [0, h]\}$. Since $F(n, 0) = 0$ for all $n \in \mathbb{R}_+$, then it easy to see that $\mu(n) = 0$ for all $n \in \mathbb{R}_+$.

Let $x = (k, n) \in \mathbb{R}_+^2$ and $n > 0$. Since the dynamical system (\mathbb{R}_+^2, T) is compactly dissipative and its Levinson center $J = \cup\{J_n \mid 0 \leq n \leq h\}$, then

$\omega_x \subseteq J$. Let $\tilde{x} = (\tilde{k}, \tilde{n}) \in \omega_x$, then there exists $t_m \rightarrow +\infty$ ($t_m \in \mathbb{Z}_+$) such that $T^{t_m}(k, n) \rightarrow (\tilde{k}, \tilde{n})$. It is evident that $\tilde{k} = 0$. Since $\lim_{t \rightarrow +\infty} T_2^t n = h$ for all $n > 0$ we obtain $\tilde{n} = h$, i.e. $\tilde{x} = (0, h)$.

Now we will prove that the fixed point $(0, h)$ is stable. If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $\delta_l \rightarrow 0$, $x_l := (k_l, n_l) \rightarrow (0, h)$ and $t_l \rightarrow +\infty$ (as $l \rightarrow +\infty$) such that $\rho(x_l, (0, h)) < \delta_l$ and

$$\rho(T^{t_l} x_l, (0, h)) \geq \varepsilon_0, \quad (24)$$

where $\rho(\cdot, \cdot)$ is the distance in \mathbb{R}_+^2 . Since $T^{t_l} x_l = (\varphi(t_l, k_l, n_l), T_2^{t_l} n_l)$, where $\varphi(t, k, n)$ is the solution of equation (23) with initial condition $\varphi(0, k, n) = k$, and $n_l \rightarrow h$ by asymptotic stability of fixed point $h \in \mathbb{R}_+$ of dynamical system (\mathbb{R}_+, T_2) we have $T_2^{t_l} n_l \rightarrow h$ as $l \rightarrow +\infty$. On the other hand by Theorem 5.2 we obtain

$$|\varphi(t_l, k_l, n_l) - \mu(T_2^{t_l})| \leq \mathcal{N} \nu^{t_l} |k_l - \mu(n_l)| = \mathcal{N} \nu^{t_l} |k_l| \rightarrow 0 \quad (25)$$

because $0 < \nu < 1$, $|k_l| \rightarrow 0$ and $t_l \rightarrow +\infty$. Taking into account that $\mu(n) = 0$ for all $n \geq 0$ we obtain $\mu(T_2^{t_l}) = 0$ for all $l \in \mathbb{N}$ and, consequently, $|\varphi(t_l, k_l, n_l)| \rightarrow 0$ as $l \rightarrow +\infty$, i.e.

$$\rho(T^{t_l} x_l, (0, h)) \rightarrow 0 \quad (26)$$

as $l \rightarrow +\infty$. The relations (24) and (26) are contradictory. The obtained contradiction proves our statement. \square

When considering Theorem 6.7 it is possible to conclude that if shareholders save less than workers and the depreciation rate of capital is big enough or, if workers save less than shareholders and the elasticity of substitution between the two factors is close to zero, then the economic system will converge to the steady state $(0, h)$ that is characterized by no capital accumulation.

Let γ be a full trajectory of dynamical system (X, \mathbb{T}, π) . Denote by $\omega_\gamma = \bigcap_{t \geq 0} \bigcup_{\tau \geq t} \gamma(\tau)$ (respectively, $\alpha_\gamma = \bigcap_{t \leq 0} \bigcup_{\tau \leq t} \gamma(\tau)$).

Theorem 6.8. *Let $\rho \in (0, 1)$, $s_r < \delta$ and J be the Levinson center of dynamical system (\mathbb{R}_+^2, T) . Then following statements hold:*

- (i) J is connected;
- (ii) $J = \bigcup \{J_n \mid 0 \leq n \leq h\}$, where $J_n := I_n \times \{n\}$ and $I_n := [a_n, b_n]$ ($a_n, b_n \in \mathbb{R}_+$);
- (iii) dynamical systems (\mathbb{R}_+, T_0) and (\mathbb{R}_+, T_h) are compactly dissipative, where $T_0(k) := T(k, 0)$ and $T_h(k) := T(k, h)$ for all $k \in \mathbb{R}_+$;

- (iv) $J_0 = [a_0, b_0] \times \{0\}$ (respectively, $J_h := [a_h, b_h] \times \{h\}$) is the Levinson center of dynamical system (\mathbb{R}_+, T_0) (respectively, (\mathbb{R}_+, T_h));
- (v) there exists at least one fixed point $p_0 \in J_0$ (respectively, $p_h \in J_h$) of the dynamical system (\mathbb{R}_+, T_0) (respectively, (\mathbb{R}_+, T_h));
- (vi) for all point $x_0 := (k_0, n_0) \in J$ (with $0 < n_0 < h$) and $\gamma \in \Phi_{x_0}$ we have $\omega_\gamma \subseteq J_h$ and $\alpha_\gamma \subseteq J_0$.

Proof. Let $\rho \in (0, 1)$ and $s_r < \delta$, then by Theorem 6.6 the dynamical system (\mathbb{R}_+^2, T) is compactly dissipative. Denote by J the Levinson center of (\mathbb{R}_+^2, T) , then by Theorem 1.33 [4] the set J is connected. Note that $J = \cup\{J_n \mid 0 \leq n \leq h\}$, where $J_n = I_n \times \{n\}$ and I_n is a compact subset of \mathbb{R}_+ . According to Theorem 2.25 [4] the set I_n is connected and, consequently, there are $a_n, b_n \in \mathbb{R}_+$ such that $I_n = [a_n, b_n]$.

Since the set $\mathbb{R}_+ \times \{0\}$ (respectively, $\mathbb{R}_+ \times \{h\}$) is invariant with respect to dynamical system (\mathbb{R}_+^2, T) , then on the set $\mathbb{R}_+ \times \{0\}$ (respectively, on $\mathbb{R}_+ \times \{h\}$) is defined a compactly dissipative dynamical system (\mathbb{R}_+, T_0) (respectively, (\mathbb{R}_+, T_h)) and the set J_0 (respectively, J_h) is its Levinson center. Taking into account that T_0 (respectively, T_h) is a continuous mapping of $J_0 = [a_0, b_0] \times \{0\}$ (respectively, $J_h = [a_h, b_h] \times \{h\}$) on itself, then there exists at least one fixed point $p_0 \in J_0$ (respectively, $p_h \in J_h$) of dynamical system (\mathbb{R}_+, T_0) (respectively, (\mathbb{R}_+, T_h)).

Let $x_0 := (k_0, n_0) \in J$ (with $0 < n_0 < h$), $\gamma \in \Phi_{x_0}$ and $x = (k, n) \in \omega_\gamma$ (respectively, $x \in \alpha_\gamma$). Then there exists a sequence $\{t_m\} \subseteq \mathbb{Z}$ such that $t_m \rightarrow +\infty$ (respectively, $t_m \rightarrow -\infty$) such that $\gamma(t_m) \rightarrow x$ as $m \rightarrow +\infty$. Since $x_0 = (k_0, n_0)$, $0 < n_0 < h$ and $pr_2(\gamma(t_m)) = T_2^{t_m}(n_0)$, then $\{T_1^{t_m}(n_0)\} \rightarrow h$ (respectively, $\{T_2^{t_m}(n_0)\} \rightarrow 0$) as $m \rightarrow +\infty$. On the other hand $x \in J$ and, consequently, $p_2(x) = h$ (respectively, $p_2(x) = 0$). Analogously we can prove that $\omega_{x_0} \subseteq J_h$ for all $x_0 = (k_0, n_0) \in \mathbb{R}_+^2$ with $n_0 > 0$, where ω_{x_0} is the ω -limit set of point x_0 . \square

References

- [1] Brianzoni S., Mammana C. and Michetti E., Complex dynamics in the neo-classical growth model with differential savings and non-constant labor force growth. *Studies in nonlinear dynamics and econometrics* 2007 (forthcoming).
- [2] Beverton R.J.H., Holt S.J., On the dynamics of exploited fish populations. *Fishery Investigations* 19, 1-533, 1957.

- [3] Bohm V., Kaas L., Differential savings, factor shares, and endogenous growth cycles. *Journal of Economic Dynamics and Control* 24, 965-980, 2000.
- [4] Cheban D. N., *Global Attractors of Nonautonomous Dissipative Dynamical Systems. Interdisciplinary Mathematical Sciences 1*. River Edge, NJ: World Scientific, 2004, 528pp.
- [5] Cheban D. N. and Mammana C., Invariant Manifolds, Global Attractors and Almost Periodic Solutions of Non-autonomous Difference equations. *Nonlinear Analyses*, serie A, v.56, No.4, 2004, pp.465-484.
- [6] Cushing J. M. and Henson S. M., Global dynamics of some periodically forced, monotone difference equations. *Journal of Difference Equations and Applications* 7, 2001, pp.859-872.
- [7] Cushing J. M. and Henson S. M., A periodically forced Beverton-Holt equation. *Journal of Difference Equations and Applications*, 8(12), 2002, pp.119-1120.
- [8] Sell G. R., *Topological Dynamics and Ordinary Differential Equations*. Van Nostrand-Reinhold, London, 1971.
- [9] Solow R.M., A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70, 1956, pp. 65-94.