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Chaos in a class of maps on the interval: the case of a small open economy with credit constraint

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Abstract

In this work we prove chaotic properties for a class of unidimensional continuous family map presenting a unique turning point and having some properties when increasing the parameter value. This set (F–function) is not conjugate to the tent map, furthermore it is not stretchable so we cannot use the well-known results about complex dynamics for these functions. However, we prove that the F–function set is chaotic in the Li–Yorke sense for a given value of the parameter onwards. We also apply the results obtained to the study of the dynamics exhibited by the economic model describing a small open economy subject to credit constraint due to moral hazard problems presented in [3]. A key role is played by the degree of financial development achieved by the economy, in fact we prove that complex behaviour can be exhibited at high level of financial development.

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1 Introduction

Li and Yorke overshoot conditions (see [6]) give analytical conditions to conclude that a map has chaotic properties, that is it is chaotic in the Li–Yorke sense. In [4], Proposition 7.4, it is shown how Li–Yorke conditions hold for a class of stretchable maps.

In this work we study a class of non-stretchable continuous unimodal family maps on the interval depending on a parameter and we prove that this system has chaotic properties in the Li–Yorke sense for sufficiently high values of the parameter.

Notice that this type of chaotic behavior does not exclude that the observable motion is indeed regular since the Li–Yorke Theorem (see [6]) implies the existence of a Scrambled set of initial points with aperiodic orbits and sensitively dependence on initial condition. However, it does not say anything on the size of this set (about the properties of the Li–Yorke chaos see, for instance, [4, 7]).

The results obtained are applied while studying an economic model describing a small open economy facing moral hazard problems of the kind proposed in [3]. We prove that economies at high level of financial development may exhibit fluctuations and complex behaviour as also discussed in [3] where, only using numerical simulations, it has been argued that there is not a level of financial development above which stability holds in the long run.

The paper is organized as follows. In Section 2 we review some basic notions and results useful for the analysis; we also define the class of the F–function whose chaotic properties are proved in this paper. In Section 3 we prove that the F–function set is chaotic in Li–Yorke sense from a given value of the parameter on. Finally we present the economic model in Section 4.

2 Preliminaries

Let $M \subseteq \mathbb{R}$ be a parameter space and define the one-parameter unidimensional family system (f_μ, I) by $f_\mu(x) \equiv f(x, \mu)$ where $f(x, \mu)$ is a continuous real function defined on $I = [0, 1]$ and dependent on the parameter $\mu \in M$.

Now we consider $f_\mu(x)$ as defined above and we give the following definitions.

Definition 2.1. *Recall [8] that a point $c_\mu \in (0, 1)$ is a turning point of $f_\mu(x)$ if there exists an open neighborhood U of c_μ such that the map $f_\mu(x)$ is strictly*

increasing on one component of $U \setminus \{c_\mu\}$ and strictly decreasing on the other one, $\forall \mu \in M$.

Definition 2.2. *We say that map $f_\mu(x)$ is unimodal in I if it has a unique turning point, $\forall \mu \in M$.*

Definition 2.3. *The set I is f_μ–invariant if $f_\mu(I) = I$, $\forall \mu \in M$.*

Now we define the following F-function set.

Definition 2.4. *Let F be the set of unimodal continuous functions $f_\mu(x) :$ $I \to I$, having the following properties.*

(i) I is f_μ–invariant;

(ii) $f_\mu(x)$ is decreasing on $I_l = [0, c_\mu)$ and strictly increasing on $I_r = (c_\mu, 1]$, $\forall \mu \in M$;

(iii) $\lim\limits_{\mu \to \sup M} c_\mu = 0$ and $\inf\limits_{\mu \in M} \{f_\mu(1)\} = k > 0$;

(iv) $f_\mu(x) < x, \forall x \in I_r$.

Notice that the properties we placed on the class of the F–functions are not so strict.

Firstly, although we are studying functions $f_\mu(x)$ invariant on the interval $[0, 1]$, property (i), similarly we can also consider the larger class of invariant functions on a given closed set $D \subset \mathbb{R}_+$ because the two maps are topologically conjugate so they exhibit the same dynamic behavior.

Secondly, with property (ii) we are considering an unimodal function having a minimum, however our considerations also hold for unimodal maps having a maximum and defined on $D \subset \mathbb{R}_-$.

Furthermore, in order to study the dynamic properties of the F–function class, a previous consideration is that we cannot use the well–known results about chaos properties of tent map (see [9], chapter 2) nor those about stretchable maps (see [4]). In fact the F–function set is not topologically conjugate to the tent map because of $f_\mu(1) \neq 1$, property (iv), and it is not stretchable because we are not assuming $f_\mu(x) \equiv \mu f_1(x)$ while we are placing property (iii).

3   Chaotic properties of the F-function set.

Theorem 3.1. *Let $f_\mu(x)$ be an F–function. Then there exists a value of μ, say μ^M, such that $f_\mu(x)$ has chaos properties for all $\mu \geq \mu^M$.*

We proceed to prove Theorem 3.1 in several steps formulated in a series of propositions.

Proposition 3.2. *Let $f_\mu(x)$ be an F–function. Then*

(a) $f_\mu(c_\mu) = 0$ *and* $f_\mu(0) = 1$, $\forall \mu \in M$;

(b) $f_\mu(x)$ *has a unique positive fixed point* $x_\mu^\star$ *such that* $x_\mu^\star \in (0, c_\mu)$;

(c) $\lim\limits_{\mu \to \sup M} x_\mu^\star = 0$.

Proof.   (a) Set I f_μ–invariant implies that $f_\mu(I) = I$ so it must be $f_\mu(c_\mu) = 0$ because c_μ is the unique minimum point of $f_\mu(x)$. Furthermore it must be $f_\mu(0) = 1$ because $f_\mu(x) < x, \forall x \in (c_\mu, 1]$ and $f_\mu(x)$ is decreasing on $I_l = [0, c_\mu)$.

(b) Let $\Phi_\mu(x) = f_\mu(x) - x$. Then $\Phi_\mu(0) = 1 > 0$ while $\Phi_\mu(c_\mu) = -c_\mu < 0, \forall \mu \in M$ (see part (a)) so $f_\mu(x)$ has at least one fixed point $x_\mu^\star$ such that $x_\mu^\star \in (0, c_\mu)$. This point must be unique because $f_\mu(x)$ is decreasing on $[0, c_\mu)$. Finally $f_\mu(x)$ has no other fixed point in $(c_\mu, 1]$ because $f_\mu(x) < x, \forall x \in (c_\mu, 1]$.

(c) Consider that $c_\mu > 0$ and $\lim\limits_{\mu \to \sup M} c_\mu = 0$. Furthermore $x_\mu^\star$ is also positive and $x_\mu^\star < c_\mu$ as proved in part (b), so it must be $\lim\limits_{\mu \to \sup M} x_\mu^\star = 0$.   □

We now consider the following result.

Proposition 3.3. *Let $f_\mu(x)$ be an F–function. Then one of the following statements is true* $\forall \mu \in M$:

(a) $0 < f_\mu(1) < x_\mu^\star < c_\mu$ *or*

(b) $0 < x_\mu^\star \leq f_\mu(1) < c_\mu$ *or*

(c) $0 < x_\mu^\star < c_\mu \leq f_\mu(1)$.

Furthermore it does exist a value of μ, say μ^M, such that statement (c) is true $\forall \mu \geq \mu^M$.

Proof. As we proved in Proposition 3.2, $0 < x_\mu^\star < c_\mu$ and also $f_\mu(1) > 0$ because $f_\mu(c_\mu) = 0$ and $f_\mu(x)$ is strictly increasing on I_r. Then one of the three statements must hold. Furthermore $\lim\limits_{\mu \to \sup M} x_\mu^\star = \lim\limits_{\mu \to \sup M} c_\mu = 0$, as we proved in Proposition 3.2, while $\inf\limits_{\mu \in M} \{f_\mu(1)\} = k > 0$ so it must exist a $\mu^M \in M$ such that statement (c) is true $\forall \mu \geq \mu^M$.   □

Proposition 3.4. *Let $f_\mu(x)$ be an F–function and let $\mu \geq \mu^M$. Then it does exist at least one pre-image of c_μ, say c_μ^{-1} with $f_\mu(c_\mu^{-1}) = c_\mu$, such that $c_\mu^{-1} \in (c_\mu, 1]$.*

Proof. Suppose $\mu \geq \mu^M$, then $0 < x_\mu^\star < c_\mu \leq f_\mu(1)$ (see Proposition 3.3). Define $\Psi_\mu(x) = f_\mu(x) - c_\mu$ then $\Psi_\mu(c_\mu) = 0 - c_\mu < 0$ while $\Psi_\mu(1) = f_\mu(1) - c_\mu \geq 0$, $\forall \mu \geq \mu^M$. If $\Psi_\mu(1) = 0$ then $f_\mu(1) = c_\mu$ so $c_\mu^{-1} = 1$ and the existence is proved, otherwise, if $\Psi_\mu(1) > 0$ then it does exist at least one $c_\mu^{-1} \in (c_\mu, 1)$ such that $\Psi_\mu(c_\mu^{-1}) = 0$ that is $f_\mu(c_\mu^{-1}) = c_\mu$. $\qquad\square$

Recall [6], let (f, I) a closed continuous system and suppose there exists a point $a \in I$ such that $f^3(a) \geq a > f(a) > f^2(a)$ then there exists a cycle of every order $n = 1, 2, 3, ...$ in I (see also [4], chapter 5) and, furthermore, (f, I) has chaos properties (see [4], chapter 7). A point a satisfying the previous inequality (label Li–Yorke overshoot condition) is called a Li–Yorke point.

Proposition 3.5. *Let $f_\mu(x)$ be an F–function and let $\mu \geq \mu^M$. Then the point c_μ^{-1} is a Li–Yorke point.*

Proof. Let c_μ^{-1} be the point we found out in Proposition 3.4. By computing a finite history of c_μ^{-1} we have that $f_\mu(c_\mu^{-1}) = c_\mu$, $f_\mu^2(c_\mu^{-1}) = f_\mu(c_\mu) = 0$ and, finally, $f_\mu^3(c_\mu^{-1}) = f_\mu(0) = 1$. Furthermore $f_\mu^3(c_\mu^{-1}) \geq c_\mu^{-1} > f_\mu(c_\mu^{-1}) > f_\mu^2(c_\mu^{-1})$ so the Li–Yorke overshoot condition is verified and c_μ^{-1} is a Li–Yorke point. Notice that if $\mu = \mu^M$ then $c_\mu^{-1} = 1$ and the map has a cycle–3. $\qquad\square$

Since $f_\mu(x)$ has a Li–Yorke point $\forall \mu \geq \mu^M$, then we can conclude that $f_\mu(x)$ has chaos properties $\forall \mu \geq \mu^M$ and Theorem 3.1 is proved.

4 Chaos in a credit constrained open economy

In this Section we use the results previously proven to demonstrate the chaotic properties of an F–function describing a small, open economy that is subject to credit constraint due to moral hazard problems, where a key role is played by the degree of financial development achieved. This model represents a reviewed version of the one considered in [1] by using a Cobb-Douglas technology. We briefly describe such a model; for further details about its construction see [3].

Consider a small open economy with an exchange good produced by a capital (K) and a country–specific input (like land, real estate or a non-tradeable natural resource), whose supply (Z) is constant and whose price

(p) is expressed in terms of units of tradeable good. The tradeable good can be consumed or accumulated as productive capital for the production in the next period.

Entrepreneurs are fund–takers who invest in production or in the international capital market, while consumers lend a fixed portion, $(1 - \alpha)$ of their wealth to the entrepreneurs or invest in the international capital market at an international equilibrium interest rate (r).

Considering a Cobb–Douglas technology, the output produced at time t is given by $y = AK^\rho z^{1-\rho}$, $\rho \in (0, 1)$, where z is the amount of the country-specific input used in period t, while $A > r$ is the total productivity factor. We assume that capital fully depreciates after one period.

The total investment I in period t is devoted to purchase both capital and country specific input. For a given level of investment, the optimal demand z arises from the maximization of the profit function subject to the budget constraint $I = K + pz$. The first order condition of the previous problem yields the following demand $z = \left(\frac{1-\rho}{p} \right) I$, therefore, the country specific input equilibrium price is obtained by equating the country specific input equilibrium demand with its constant supply Z. Finally, we may write the total equilibrium output y in terms of the level of investment I and the price p, $y = G(p)I$, with $G(p) = \frac{A\rho^\rho (1-\rho)^{1-\rho}}{p^{1-\rho}}$. Note that $G(p)$ can be viewed as the gross return of a unit of investment.

Within the model, a key role is played by financial factors as a source of instability due to the presence of moral–hazard imperfections in the credit market such that an entrepreneur can borrow each time at best an amount L that is proportionate to its wealth W in the early stage, that is $L \leq \mu W$ (this hypothesis is largely discussed in [2] where it is assumed that the entrepreneur's wealth serves as a collateral for the loan). Coefficient μ can be considered as an indicator of the financial development level of the national economy (about this relation see [5]). Therefore, with a credit constraint being present, the maximum investment amount is $I = (1 + \mu)W$.

At time t entrepreneurs borrow, invest and produce by bearing the costs of the productive factor used; then they make profits and pay off their debt to lenders at an interest rate r while agents use up and save, thus determining the amount of wealth available to entrepreneurs in the following period. The dynamical law of wealth evolution is therefore given by $f(W) = (1 - \alpha)(y - rL)$.

Let us now consider the following two cases.

- If $G(p) \geq r$ the return of the productive investment is higher then the one in the capital market, so entrepreneurs invest in production the maximum amount they can borrow considering the credit con-

straint, therefore $I = (1 + \mu)W$. By replacing this relation with the one determining the equilibrium price of Z with the maximized production relation we obtain $y = \xi W^\rho$ where $\xi = A\rho^\rho (1 + \mu)^\rho Z^{1-\rho}$. The dynamical equation of wealth will be therefore given by $f_\mu^l(W) = (1-\alpha)W^\rho \left(\xi - r\mu W^{1-\rho}\right)$ that holds if $G(p) \geq r$ that is $W \leq W_\mu^m$ where $W_\mu^m = \left(\frac{A}{r}\right)^{\frac{1}{1-\rho}} \frac{Z}{1+\mu} \rho^{\frac{\rho}{1-\rho}}$.

- If $G(p) < r$ entrepreneurs invest in production only as long as the production investment return equals the one in the capital market, that is $y - rL = rW$ which, when replaced in the dynamical equation of wealth, results in $f_\mu^r(W) = (1-\alpha)rW$ that holds for $W > W_\mu^m$.

The final model describing the time evolution of the borrowers' wealth status $(W \in \mathbb{R}_+)$ is given by the following map:

$$f_\mu(W) = \begin{cases} f_\mu^l(W) = (1-\alpha)W^\rho \left(\xi - r\mu W^{1-\rho}\right), & 0 \leq W < W_\mu^m; \\ f_\mu^r(W) = (1-\alpha)rW, & W \geq W_\mu^m. \end{cases} \quad (1)$$

where $W_\mu^m = \left(\frac{A}{r}\right)^{\frac{1}{1-\rho}} \frac{Z}{1+\mu} \rho^{\frac{\rho}{1-\rho}}$ and $\xi = A\rho^\rho (1 + \mu)^\rho Z^{1-\rho}$.[1]

Here $\alpha \in (0,1)$ is the consumption rate so $(1-\alpha)$ is the constant fraction the consumers save of their own wealth; $A > r$ is the total productivity factor, where $r > 1$ is the constant equilibrium rate in the international capital market; finally $\rho \in (0,1)$ is the marginal rate of substitution between the productivity factors while $Z \in \mathbb{R}_+$ is the country specific production factor which we assume to be constant. From economic considerations it is also assumed that $(1-\alpha)r < 1$.

Since the investment at each period is upper bound by $(1 + \mu)W$, then the proportional coefficient $\mu \geq 0$ can be understood as a credit multiplier that reflects the level of the financial development in the domestic economy. We are interested in studying the dynamics exhibited by (1) when varying the parameter μ.

It is straightforward to see that system (1) is continuous.

First we prove the following Proposition.

Proposition 4.1. *Let $f_\mu(W)$ given by (1) and let $\rho^{\frac{1}{1-\rho}} < (1-\alpha)r$. Then there exists a value of μ, say $\bar{\mu} > 0$, such that $f_\mu(W)$ is unimodal in the set $S = \left[f_\mu(W_\mu^m), f_\mu^2(W_\mu^m)\right], \forall \mu > \bar{\mu}$, with W_μ^m being its unique turning–point.*

Proof. In order to prove the previous Proposition 4.1, we proceed by showing that there exists a $\bar{\mu}$ such that the following properties

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[1]This is the model presented in [3] while assuming no exogenous income in terms of tradeable good

(a) W_μ^m is a turning–point of $f_\mu(W)$;

(b) $W_\mu^m \in S' = \left(f_\mu(W_\mu^m), f_\mu^2(W_\mu^m)\right)$;

(c) $W_\mu \in \mathcal{C}_{\mathbb{R}^+}(S')$, where W_μ^M is the maximum of $f_\mu^l(W)$;

are all verified $\forall \mu > \bar{\mu}$.

(a) It is straightforward to verify that $f_\mu^l(W)$ is a strictly concave function having a maxima in $W_\mu^M = \left(\frac{r\mu}{\xi\rho}\right)^{\frac{1}{\rho-1}} \geq 0$, $\forall \mu \in [0, +\infty)$. On the other hand $f_\mu^r(W)$ is a linear increasing function, $\forall \mu \in [0, +\infty)$. Then W_μ^m is a turning–point of $f_\mu(W) \Leftrightarrow f_\mu^l(W)$ is strictly decreasing in a left–neighborhood of $W_\mu^m \Leftrightarrow W_\mu^m > W_\mu^M$. Easily we have

$$W_\mu^M - W_\mu^m = \left(\frac{r\mu}{\xi\rho}\right)^{\frac{1}{\rho-1}} - \left(\frac{A}{r}\right)^{\frac{1}{1-\rho}} \frac{Z}{1+\mu} \rho^{\frac{\rho}{1-\rho}} =$$

$$= Z\left(\frac{r}{A}\right)^{\frac{1}{\rho-1}} \frac{\left[\mu^{\frac{1}{\rho-1}}\rho^{\frac{\rho+1}{1-\rho}}(1+\mu)^{\frac{\rho}{1-\rho}} + \mu^{\frac{\rho}{\rho-1}}\rho^{\frac{\rho+1}{1-\rho}}(1+\mu)^{\frac{\rho}{1-\rho}} - \rho^{\frac{\rho}{1-\rho}}\right]}{1+\mu} =$$

$$= Z\left(\frac{r}{A}\right)^{\frac{1}{\rho-1}} (\rho(1+\mu))^{\frac{\rho}{1-\rho}}(1+\mu)\left[\mu^{\frac{1}{\rho-1}}\rho^{\frac{1}{1-\rho}} - (1+\mu)^{\frac{1}{\rho-1}}\right]$$

then $W_\mu^M - W_\mu^m < 0$ iff $\mu > \frac{\rho}{1-\rho} = \mu_1$ so W_μ^m is a turning–point.

(b) In order to have $W_\mu^m \in S'$, first notice that $f_\mu(W_\mu^m) = (1-\alpha)rW_\mu^m < W_\mu^m$, $\forall \mu \geq 0$, because we are assuming $(1-\alpha)r < 1$. So we only need to verify that $f_\mu^2(W_\mu^m) > W_\mu^m$. Let $q(\mu) = f_\mu^2(W_\mu^m) - W_\mu^m$ then

$$q(\mu) = \left(\frac{A}{r}\right)^{\frac{1}{1-\rho}} Z\rho^{\frac{\rho}{1-\rho}}$$

$$\left\{(1-\alpha)^2 r\left[AZ^{1-\rho}\rho^\rho(1+\mu)^{\rho-1}\left(1 - \frac{\mu}{1+\mu}((1-\alpha)r)^{1-\rho}\right)\right] - \frac{1}{1+\mu}\right\}$$

and it is trivial to verify that $\lim_{\mu \to \infty} q(\mu) = 0^+$ so it must exist a μ_2 such that $q(\mu) > 0$, $\forall \mu > \mu_2$. So we can conclude that W_μ^m belongs to S', $\forall \mu > \mu_2$.

(c) In order to have $W_\mu^M \notin S'$, a sufficient condition is that $W_\mu^M < f_\mu(W_\mu^m)$.

Let $h(\mu) = W_\mu^M - f_\mu(W_\mu^m)$. Easily we have

$$h(\mu) = \left(\frac{r\mu}{\xi\rho}\right)^{\frac{1}{\rho-1}} - (1-\alpha)r\left(\frac{A}{r}\right)^{\frac{1}{1-\rho}}\frac{Z}{1+\mu}\rho^{\frac{\rho}{1-\rho}} =$$

$$= Z\left(\frac{A}{r}\right)^{\frac{1}{1-\rho}}\frac{\rho^{\frac{\rho}{1-\rho}}}{1+\mu}\left[\mu^{\frac{1}{\rho-1}}\rho^{\frac{1}{1-\rho}}(1+\mu)^{\frac{1}{1-\rho}} - (1-\alpha)r\right]$$

In order to have $h(\mu) < 0$, the inequality $\frac{\mu^{\frac{1}{\rho-1}}}{(1+\mu)^{\frac{1}{\rho-1}}} < \frac{(1-\alpha)r}{\rho^{\frac{1}{1-\rho}}}$ must hold.

Since $\lim\limits_{\mu\to\infty}\frac{\mu^{\frac{1}{\rho-1}}}{1+\mu^{\frac{1}{\rho-1}}} = 1$ then if $\rho^{\frac{1}{1-\rho}} < (1-\alpha)r$ (that is verified for economically suitable values of the parameters), the inequality holds $\forall\mu > \mu_3$.

Finally, let $\bar\mu = \max\{\mu_1, \mu_2, \mu_3\}$ then statements (a) and (b) and (c) hold $\forall\mu > \bar\mu$ and Proposition 4.1 is proved. $\square$
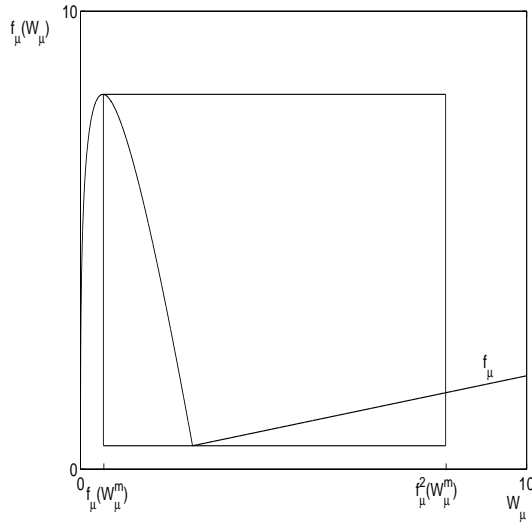

Figure 1: The invariant set owned by f_μ as defined in (1) for a sufficiently high value of μ. In the simulation we choose $\mu = 40$ while $\bar\mu \simeq 25.24$.

Corollary 4.2. *Set S is $f_\mu(W)$–invariant, $\forall\mu > \bar\mu$.*

The proof is trivial and left to the reader.

In figure 1 the invariant set S is pictured for a sufficiently high value of μ. In the following simulations we fix all the parameters but μ at economically plausible values ($\alpha = 0.8$, $\rho = \frac{1}{3}$, $r = 1.02$, $A = 1.5$ and $Z = 100$).

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Now we consider the case of $\mu > \bar{\mu}$. Notice that this case can be studied in a simpler way if we compose function $f_\mu(W) : S \to S$ by an homeomorphism h in such away $g_\mu(x) = h(f(h^{-1})) : [0,1] \to [0,1]$. Of course the dynamics of f_μ and g_μ are conjugate, that is, the two functions have the same geometric properties (notice that function h is simply a linear transformation depending on μ that sends $f_\mu(W_\mu^m)$ to 0 and $f_\mu^2(W_\mu^m)$ to 1, for each μ) and they exhibit the same dynamics. The advantage of working with g_μ instead of f_μ lies on the fact that the windows where the asymptotic dynamics occurs are unchanging with μ (namely $[0,1]$) hence, the static comparative among different values of μ is much easier. If we denote by g_μ^l and g_μ^r the two piecewise components (left and right respectively), then g_μ is defined as follow:

$$g_\mu(x) = \begin{cases} g_\mu^l(x), & 0 \le x < x_\mu^m; \\ g_\mu^r(x), & x_\mu^m \le x \le 1. \end{cases} \tag{2}$$

where $g_\mu^l(x)$ is a monotone decreasing concave function while $g_\mu^r(x)$ is a linear function with constant slope $(1-\alpha)r$. Then it is straightforward to prove the following proposition.
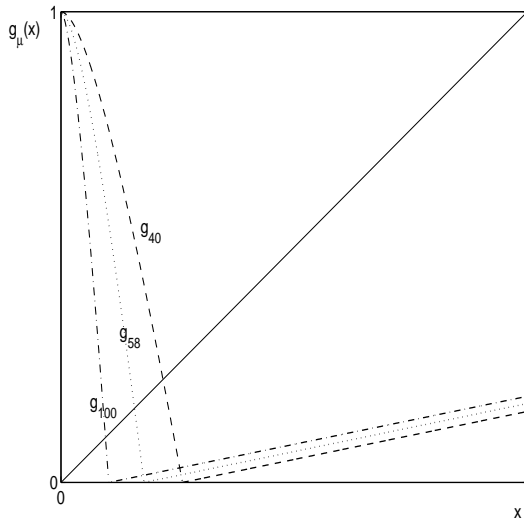

Figure 2: Scheme of g_μ in the cases of: $g_\mu(1) \le x_\mu^\star < x_\mu^m$ when $\mu = 40$; $x_\mu^\star < g_\mu(1) < x_\mu^m$ when $\mu = 58$ and $x_\mu^\star < x_\mu^m \le g_\mu(1)$ when $\mu = 100$. Some computations show that, given the other parameters values we choose, $\mu^m \simeq 54.925$ while $\mu^M \simeq 60.936$.

Proposition 4.3. *Let $g_\mu(x)$ given by (2). Then $g_\mu(x)$ is an F–function.*

Proof. Since g_μ has in $[0,1]$ the same geometric properties that f_μ has in S, then the following considerations hold.

(i) $g_\mu(I)$ goes from the image of the minimum point that is $g_\mu(x_\mu^m) = 0$ up to the maximum between $g_\mu(0) = 1$ and $g_\mu(1) < (1-\alpha)r < 1$, $\forall\mu$. So $g_\mu(I) = I$ and I is g_μ–invariant;

(ii) As we proved in Proposition 4.1, if $\mu > \bar\mu$ then $f_\mu(W)$ is decreasing in $[f_\mu(W_\mu^m), W_\mu^m)$ and strictly increasing on $(W_\mu^m, f_\mu^2(W_\mu^m)]$ so $g_\mu(x)$ is decreasing on $I_l = [0, x_\mu^m)$ and strictly increasing on $I_r = (x_\mu^m, 1]$, $\forall\mu \in M = (\bar\mu, +\infty)$;

(iii) $\lim\limits_{\mu\to\sup M} x_\mu^m = \lim\limits_{\mu\to\sup M} W_\mu^m = 0$ and $\inf\limits_{\mu\in M} g_\mu(1) > 0$;

(iv) $g_\mu(x) = (1-\alpha)rx < x$, because of $(1-\alpha)r < 1$, $\forall x \in I_r$.

$\square$

Since g_μ is an F-function, we can conclude that it does exist a μ^M such that the map has chaos properties for all $\mu \geq \mu^M$ as we proved in Theorem 3.1. Furthermore, considering that both the fixed point $x_\mu^\star$ and the turning point x_μ^m are strictly monotone decreasing functions of μ while $g_\mu(1)$ is a strictly monotone increasing function of μ, it can be proven that there exist two values of μ, say $\mu^M > \mu^m > \bar\mu$ such that:

- $g_\mu(1) \leq x_\mu^\star < x_\mu^m, \forall\mu \in (\bar\mu, \mu^m]$;

- $x_\mu^\star < g_\mu(1) < x_\mu^m, \forall\mu \in (\mu^m, \mu^M)$;

- $x_\mu^\star < x_\mu^m \leq g_\mu(1), \forall\mu \in [\mu^M, +\infty)$.

In figure 2 these three cases are presented for different given values of μ.

Finally, as postulated by Theorem 3.1 here proven, $\forall\mu \in [\mu^M, +\infty)$ a Li–Yorke point does exist. In fact, let $y_\mu = (x_\mu^m)^{-1} > x_\mu^m$ then y_μ is a Li–Yorke point as pictured in Figure 3.

Because of the existence of this point, there exists a cycle of every order in $[0,1]$ and the Li–Yorke Theorem [6] also holds so the map has chaos properties (about such properties see [4]).

The Theorem we demonstrated proves chaos properties for economies that are sufficiently developed in financial terms. It means that there is no sufficiently high level of financial development to guarantee stability in small economies facing moral hazard problems according to the results reached in [3] using numerical simulations.
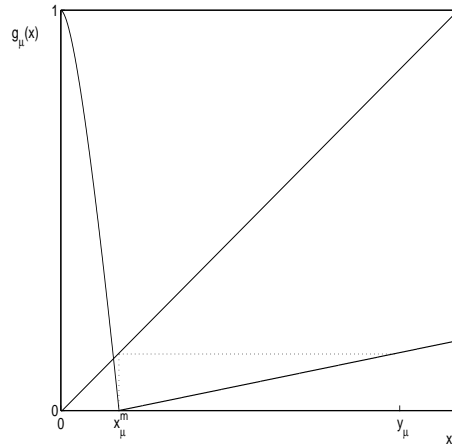
Figure 3: Existence of the Li–Yorke point $y_\mu = (x_\mu^m)^{-1}$. Here $\mu = 70 > \mu^M$.

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